Quiz 4 Name: SOLUTIONS

Math 126: Calculus II
21 October 2019

1. Write $0.3939\overline{39}$ as a geometric series. Then find the sum of this geometric series as a ratio of integers (i.e., a fraction with integer numertor and integer denominator).

$$0.3939\overline{39} = 0.39 + 0.0039 + 0.000039 + 0.00000039 + \cdots$$

This is a geometric series with $a = \frac{39}{100}$ and $r = \frac{1}{100}$.

Sum is: $$\frac{a}{1-r} = \frac{\frac{39}{100}}{1-\frac{1}{100}} = \frac{\frac{39}{100}}{\frac{99}{100}} = \frac{39}{99} = \frac{13}{33}.$$

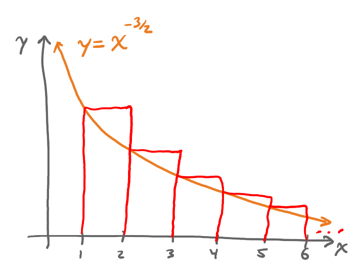
2. Does $\displaystyle\sum_{n=1}^{\infty} n^{-3/2}$ converge or diverge? (*Hint*: integral test!)

The sum $\displaystyle\sum_{n=1}^{\infty} n^{-3/2}$ converges if and only if $\displaystyle\int_1^{\infty} x^{-3/2}\,dx$ converges.

$$\int_1^{\infty} x^{-3/2}\,dx = \lim_{b\to\infty}\int_1^b x^{-3/2}\,dx = \lim_{b\to\infty}\Big[-2x^{-1/2}\Big]_{x=1}^{x=b} = \lim_{b\to\infty}\left[\frac{-2}{\sqrt{b}} - \frac{-2}{\sqrt{1}}\right] = 2$$

Since the integral converges, so does the sum.

Note that the sum doesn't converge to 2.
The sum equals a number slightly larger than 2.



St. Olaf Honor Pledge: I pledge my honor that on this examination I have neither given nor received assistance not explicitly approved by the professor and that I have seen no dishonest work.

Signed: ______________________

☐ I have intentionally not signed the pledge. (Check the box if appropriate.)

Taylor Series

So far, we have found power series that converge to functions such as

$$\frac{1}{1-x}, \quad \frac{1}{1-3x}, \quad \frac{4}{1-x^2}, \quad \text{and} \quad \frac{1}{(1-x)^2}.$$

Can we pick *any* function $f(x)$ and find a power series that converges to it?

1. Let's try $f(x) = \sin(x)$. We want to find numbers $c_0, c_1, c_2, c_3, \ldots$ such that

$$\sin(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \qquad (*)$$

(a) Set $x = 0$ in equation $(*)$. What does this tell you about the value of c_0?

$\sin(0) = c_0 + c_1(0) + c_2(0)^2 + c_3(0)^3 + \cdots$

$\boxed{0 = c_0}$

(b) Write down the derivative of equation $(*)$. Then set $x = 0$. Does this tell you the value of c_1?

differentiate

$\cos(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$

$\cos(0) = c_1 + 2c_2(0) + 3c_3(0)^2 + 4c_4(0)^3 + \cdots$

$\boxed{1 = c_1}$

(c) Now write down the *second* derivative of equation $(*)$. Then set $x = 0$. What do you find?

differentiate

$-\sin(x) = 2c_2 + 6c_3 x + 12c_4 x^2 + \cdots$

$-\sin(0) = 2c_2 + 6c_3(0) + 12c_4(0)^2 + \cdots$

$0 = 2c_2$ so $\boxed{c_2 = 0}$

(d) How can you find the value of c_3?

differentiate:

$-\cos(x) = 6c_3 + 24c_4 x + 60c_5 x^2 + \cdots$

$-\cos(0) = 6c_3 + 24c_4(0) + 60c_5(0)^2 + \cdots$

$-1 = 6c_3$ so $\boxed{c_3 = \frac{-1}{6}}$

(e) How can you find the value of c_4? How about c_5?

Do you see a pattern?

Fourth deriv, plug in $x=0$, find $\boxed{c_4 = 0.}$

Fifth deriv, plug in $x=0$, find $\boxed{c_5 = \frac{1}{120}}$ $\left(\cos(0) = 120c_5\right)$

(f) **Conjecture:** How would you find the value of c_n?

Differentiate equation $(*)$ n times.

Set $x=0$.

Solve for c_n.

TAYLOR SERIES

$\sin(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$

We found:

$\sin(x) = 0 + 1x + 0x^2 - \frac{1}{6}x^3 + 0x^4 + \frac{1}{120}x^5 + \cdots$

$\boxed{\sin(x) = x - \frac{x^3}{6} + \frac{1}{120}x^5} + \cdots$

Taylor polynomial of degree 5

2. Let's figure out the general formula for the coefficients of a power series (centered at 0) that converges to a function $f(x)$. Suppose we want:

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots$

(a) How can you find c_0 in terms of $f(x)$? How about c_1? ...c_2?

First, set $x=0$, and solve: $f(0) = c_0$.

Then: differentiate, set $x=0$, and solve for c_1. $f'(0) = c_1$

Then: differentiate again, set $x=0$, and solve for c_2: $f''(x) = 2c_2 + 6c_3 x + 12 c_4 x^2 + \cdots$

$f''(0) = 2c_2$, so $c_2 = \frac{1}{2} f''(0)$.

(b) I just heard somebody say, "If you take the derivative n times, every power of x lower than n disappears!" Do you agree with this statement? Does this help you find the value of c_n?

nth derivative: $f^{(n)}(x) = n(n-1)(n-2)\cdots(3)(2)(1)c_n + \_\_\_ x + \_\_\_ x^2 + \cdots$

$f^{(n)}(0) = n!\, c_n$

(c) Complete the following sentence:

If $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$, then $c_n = \dfrac{f^{(n)}(0)}{n!}$

3. Fill in the following chart (quickly) for $f(x) = e^x$.

n	**nth derivative of $f(x)$**	**evaluated at $x = 0$**
0	e^x	1
1	e^x	1
2	e^x	1
3	e^x	1
4	e^x	1
5	e^x	1
6	e^x	1
7	e^x	1
8	e^x	1

What power series converges to e^x? What is the radius of convergence?

$e^x = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$

Let $f(x) = e^x$

$e^x = f(0) + \frac{f'(0)}{1} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \frac{f^{(5)}(0)}{5!} x^5 + \cdots$

$$\boxed{e^x = 1 + 1x + \tfrac{1}{2} x^2 + \tfrac{1}{6} x^3 + \tfrac{1}{4!} x^4 + \tfrac{1}{5!} x^5 + \cdots}$$

Taylor series for e^x.

4. The previous problems have only considered power series centered at $x = 0$. What if we want a power series centered at $x = a$? That is,

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$$

Discuss with your table partners: What would the formula for c_n look like in this case?

Let f be a function, all of whose derivatives exist at $x = a$. The **Taylor series** for f centered at $x = a$ is the series defined by:

$$f(x) = f(a) + \frac{f'(a)}{1}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

In the special case where $a = 0$, the Taylor series is called the **Maclaurin series** for f.

5. Find the Taylor series for $f(x) = \ln(x)$ centered at $x = 1$. What is its interval of convergence?

6. Find the Maclaurin series for $f(x) = \sin(2x)$.

Can you think of *two* different ways to do this?