Improper Integrals

1. (a) **Graphing time!** Choose someone in your group to sketch the graph of $y = \dfrac{1}{x^2}$ on the wall near your table.

(b) **Warm-up:** Use the FTC to compute these progressively fun areas.

$$\int_1^{10} \frac{1}{x^2}\,dx \qquad \int_1^{100} \frac{1}{x^2}\,dx \qquad \int_1^{1000} \frac{1}{x^2}\,dx$$

☝ Draw the regions! Write out your answers as decimals numbers.

(c) Now use the FTC to find a formula for $\displaystyle\int_1^b \frac{1}{x^2}\,dx$.

☝ You can pretend that b is some number larger than 1.

(d) **Milo:** Great! That must mean that $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ equals 1.

Erez: Wait...what? Infinity *in an integral*? You can't just replace b with ∞ like that!

Ava: Yeah, ∞ is not a number.

Milo: I know it looks awful, but it works!

Group chat: What is Milo thinking? Can you make sense of $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$?

2. (a) Now use the FTC to compute the following:

$$\int_{1/10}^{1} \frac{1}{x^2}\,dx \qquad \int_{1/100}^{1} \frac{1}{x^2}\,dx \qquad \int_{1/1000}^{1} \frac{1}{x^2}\,dx$$

(b) **Group chat:** Now try to find a formula for $\displaystyle\int_a^1 \frac{1}{x^2}\,dx$.

Your formula will depend on a. You may suppose $0 < a < 1$.

(c) **Ava:** Ooooh, now I want to calculate $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$.

Erez (shaking head): You can't do that, because $\dfrac{1}{x^2}$ is not defined at $x = 0$.

Ava: True, but the integral still makes sense!

Group chat: Does $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$ make sense? What happens to $\displaystyle\int_a^1 \frac{1}{x^2}\,dx$ as a gets closer and closer to zero?

3. (a) Find a formula for $\displaystyle\int_a^1 \frac{1}{x}\,dx$.

Again, suppose $0 < a < 1$.

(b) How should we evaluate $\displaystyle\int_0^1 \frac{1}{x}\,dx$?

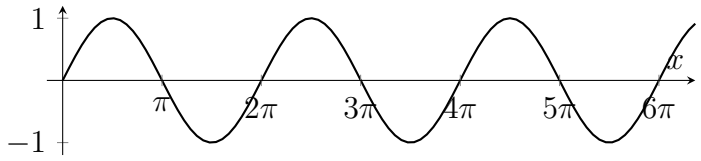
(c) How should we evaluate $\displaystyle\int_{-1}^0 \frac{1}{x}\,dx$?

(d) **Group chat:** How should we evaluate $\displaystyle\int_{-1}^1 \frac{1}{x}\,dx$?

4. Evaluate $\displaystyle\int_0^4 \frac{1}{\sqrt{x}}\,dx$. Either determine the number that it converges to, or explain why it diverges.

5. Evaluate $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$. Either determine the number that it converges to, or explain why it diverges.

6. Here is a graph of the function $f(x) = \sin(x)$:



(a) Quick! Without calculating any antiderivatives or doing any algebra, find each of the following:

$$\int_0^{2\pi} \sin(x)\ dx \qquad \int_0^{4\pi} \sin(x)\ dx \qquad \int_0^{6\pi} \sin(x)\ dx$$

(b) **Group chat:** Based on the previous question, what do you think $\displaystyle\int_0^{\infty} \sin(x)\ dx$ should equal?

(c) Use the FTC to evaluate $\displaystyle\int_0^{\pi} \sin(x)\ dx$.

This is the area of one "bump" of the graph of $\sin(x)$.

(d) Without calculating any more antiderivatives or doing any more algebra, find each of the following:

$$\int_0^{3\pi} \sin(x)\ dx \qquad \int_0^{5\pi} \sin(x)\ dx \qquad \int_0^{7\pi} \sin(x)\ dx$$

(e) **Group chat:** Based on the previous question, what do you think $\displaystyle\int_0^{\infty} \sin(x)\ dx$ should equal?

(f) **Group chat:** Considering parts (b) and (e) above, what can we say about $\displaystyle\int_0^{\infty} \sin(x)\ dx$?

7. Experiment time! Choose a few different values of p and compute $\displaystyle\int_1^{\infty} \frac{1}{x^p}\, dx$. For which values of p do you get an actual numerical answer?