MATH 126 A/B — 15 October 2025

Recall from last time:

LIMIT COMPARISON TEST

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where $0 < c < \infty$, then the series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

RATIO TEST

Calculate $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r$. Then:

- If $r < 1$, then $\sum a_n$ converges
- If $r > 1$, then $\sum a_n$ diverges.
- If $r = 1$, then this test is inconclusive

More Series

1. Determine whether each of the following series converges or diverges (with explanation).

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n+7}$

Converges!

Limit Comparison test with $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$

$$\lim_{n\to\infty} \frac{\frac{1}{2^n+7}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n \cdot \frac{1}{2^n}}{(2^n+7)\frac{1}{2^n}} = \lim_{n\to\infty} \frac{1}{1+\frac{7}{2^n}} = 1$$

Since $\sum \frac{1}{2^n}$ converges (geo. series with $r=\frac{1}{2}$), the series here also converges.

You could also use the ratio test here!

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{2^{n+1}+7}}{\frac{1}{2^n+7}}\right| = \frac{1}{2}$$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{4^n+n}{3^n}$

Diverges.

Divergence Test:

$$\lim_{n\to\infty} \frac{4^n+n}{3^n} = \underset{(\infty)}{\text{DNE}} \neq 0,$$ so this series diverges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4-n+5}$

Converges.

Limit Comparison test with $\sum \frac{1}{n^4}$:

$$\lim_{n\to\infty} \frac{\frac{1}{n^4-n+5}}{\frac{1}{n^4}} = \lim_{n\to\infty} \frac{n^4}{n^4-n+5} = 1$$

Since $\sum \frac{1}{n^4}$ converges (p-series with $p=4$), the series here converges also

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(d) $\displaystyle\sum_{n=10}^{\infty} \frac{1}{n^{0.9}-6}$

Diverges.

Limit Comparison Test with $\sum \frac{1}{n^{0.9}}$:

$$\lim_{n\to\infty} \frac{\frac{1}{n^{0.9}-6}}{\frac{1}{n^{0.9}}} = \lim_{n\to\infty} \frac{n^{0.9}}{n^{0.9}-6} = 1$$

Since $\sum \frac{1}{n^{0.9}}$ diverges (p-series with $p=0.9$), the series here also diverges.

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{n!}$

Converges.

Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{4^{n+1}}{(n+1)!}}{\frac{4^n}{n!}}\right| = \lim_{n\to\infty} \frac{4^{n+1}}{(n+1)!}\cdot\frac{n!}{4^n} = \lim_{n\to\infty} \frac{4}{n+1} = 0 < 1.$$

Thus, this series converges by the ratio test.

2. Use a calculator or computer to compute the first ten partial sums of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

Then plot these ten values on a number line. What do you notice?

The partial sums converge to a value near 0.7!

$S_1 = 1$

$S_2 = 0.5$

$S_3 = 0.8\overline{3}$

$S_4 = 0.58\overline{3}$

$S_5 = 0.78\overline{3}$

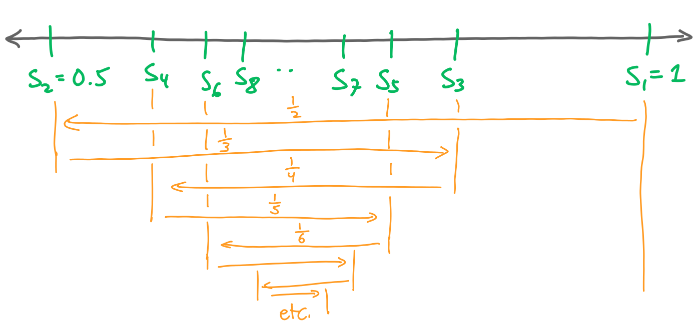
$S_6 = 0.61\overline{6}$

$S_7 = 0.7595...$

$S_8 = 0.6345...$

$S_9 = 0.7454...$

$S_{10} = 0.6456...$



ALTERNATING SERIES: Terms strictly alternate $+, -, +, -, \cdots$

Form: $\sum_{n=1}^{\infty} (-1)^n a_n$ or $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ where $a_n > 0$.

ALTERNATING SERIES TEST:

Given an alternating series $\sum_{n=1}^{\infty} (-1)^n a_k$, if the sequence $\{a_n\}$ of positive terms satisfies $\lim_{n\to\infty} a_n = 0$, then the alternating series converges.

CONDITIONAL vs. ABSOLUTE CONVERGENCE:

The series $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

(make every term positive!)

The series $\sum_{n=1}^{\infty} a_n$ converges conditionally if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

➡ Conditional convergence means that the series converges if some terms are negative, but <u>not</u> if you make every term positive!

3. Which of the following alternating series converges?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = -1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \cdots$

Do the terms go to zero?

$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, so the series converges by the alternating series test

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(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, n}{2n+3} = \frac{1}{5} - \frac{2}{7} + \frac{3}{9} - \frac{4}{11} + \frac{5}{13} - \cdots$

Since $\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2} \neq 0$, the series diverges by the divergence test

4. Determine whether each alternating series is absolutely convergent, conditionally convergent, or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, n!}{2^n}$ Since $\lim_{n\to\infty} \frac{n!}{2^n} = \underset{(\infty)}{\text{DNE}}$, this series diverges (by the divergence test).

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+5}}$ Since $\lim_{n\to\infty} \frac{1}{\sqrt{n+5}} = 0$, this series converges by the alternating series test.

However, $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt{n+5}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+5}} = \sum_{n=6}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series $(p = \frac{1}{2})$.

Therefore, this series is conditionally convergent.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1.5}}$ Since $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^{1.5}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$ is a convergent p-series $(p = 1.5)$,

the given series is absolutely convergent.

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n\, n}{n^3 - 1}$ Since $\sum_{n=2}^{\infty} \left| \frac{(-1)^n \cdot n}{n^3 - 1} \right| = \sum_{n=2}^{\infty} \frac{n}{n^3-1}$ converges

by limit comparison with $\sum_{n=2}^{\infty} \frac{1}{n^2}$,

the given series is absolutely convergent.

LIMIT COMPARISON TEST:

$$\lim_{n\to\infty} \frac{\frac{n}{n^3-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n}{n^3-1} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^3}{n^3-1} = 1$$

So both $\sum \frac{n}{n^3-1}$ and $\sum \frac{1}{n^2}$ behave the same way

5. Suppose you want to help a friend study for the next exam. Write down several infinite series that you could give your friend as practice for determine convergence or divergence? Try to write down a selection of series that together involve *all* of the series concepts we have studied so far!