MATH 126 A/B — 24 October 2025

LAST TIME: If we have a function $f(x)$ and we want a power series that converges to $f(x)$,

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots \quad \leftarrow \text{centered at } x=0$$

then we <u>need</u> $c_n = \dfrac{f^{(n)}(0)}{n!}$.

MACLAURIN SERIES FOR $f(x)$:

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Note that this series is centered at $x=0$.

If the series were instead centered at $x=a$, it would be a

TAYLOR SERIES: $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

IMPORTANT EXAMPLES:

Function	Maclauri Series	Values of x that make the series equal $f(x)$
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$	$-1 < x < 1$
$\arctan(x)$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$-1 \le x \le 1$
$\cos(x)$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	all x
$\sin(x)$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	all x
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	all x
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$	$-1 < x \le 1$

Taylor and Maclaurin Series

1. Find the sum of each series. (All the series converge and NONE of them are geometric.)

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$

Maclaurin series for e^x with $x = 2$

Hint: Look back and forth between this page and the board.

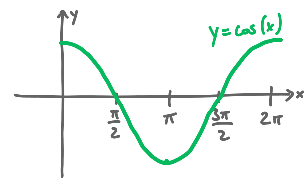
(b) $1 + (\ln 3) + \frac{(\ln 3)^2}{2!} + \frac{(\ln 3)^3}{3!} + \frac{(\ln 3)^4}{4!} + \ldots = e^{\ln 3} = 3$

Maclaurin series for e^x with $x = \ln(3)$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 9^n \pi^{2n}}{4^n (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{(3^2)^n \pi^{2n}}{(2^2)^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{3\pi}{2}\right)^{2n} = \cos\left(\frac{3\pi}{2}\right) = 0$

Maclaurin series for $\cos(x)$ with $x = \frac{3\pi}{2}$

Note that $9 = 3^2$ and $4 = 2^2$.



2. Renita: I really want to find the Maclaurin series for $f(x) = x^2 \cos(x^2)$.

Logan: OK! All we have to do is take the derivative of $f(x)$ over and over and over repeatedly forever, and then plug in $x = 0$ each step along the way.

Group chat: Why Logan's strategy a *very bad* idea?

We would need to use both the product rule and the chain rule, which would quickly make these derivatives very complicated!

Renita: I have a faster way, Logan! Instead, let's use the Maclaurin series *we already know:*

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Logan: Oh...we can do this in two steps! First, we will do the $\cos(x^2)$ piece:

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$$

Group chat: What did Logan just do? What's the next step towards finding the series for $f(x) = x^2 \cos(x^2)$?

Multiply by x^2:

$$x^2 \cdot \cos(x^2) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n} x^2}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n)!}$$

definition of Maclaurin series

NOTE: The coefficient of x^n in the Maclaurin series <u>must</u> be $\frac{f^{(n)}(0)}{n!}$.

3. Macie: Hey Logan, I am so happy we just found the Maclaurin series for $f(x) = x^2\cos(x^2)$.

$$x^2 \cos x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n)!} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Logan: Cool! I have another problem now. I want to find both $f^{(17)}(0)$ and $f^{(18)}(0)$.

Renita: Well, you definitely do NOT want to take the derivative of $f(x)$ 17 or 18 times.

Group chat: Why did Renita say this?

Taking 18 derivatives using the product rule and chain rule would be very messy!

Macie: I agree. The information I want is hidden in the Maclaurin Series!

Group chat: How can you use the Maclaurin Series for $f(x) = x^2 \cos(x^2)$ to find $f^{(17)}(0)$ and $f^{(18)}(0)$?

$$x^2\cdot\cos(x^2) = \underset{n=0}{x^2} - \underset{n=1}{\frac{x^6}{2}} + \underset{n=2}{\frac{x^{10}}{4!}} - \underset{n=3}{\frac{x^{14}}{6!}} + \underset{n=4}{\frac{x^{18}}{8!}} - \cdots$$

The series has no x^{17} term, so $0 = \frac{f^{(17)}(0)}{17!}$, and thus $f^{(17)}(0) = 0$.

Coefficient of x^{18} is $\frac{1}{8!}$, so $\frac{1}{8!} = \frac{f^{(18)}(0)}{18!}$, and thus $f^{(18)}(0) = \frac{18!}{8!}$

We didn't do this in class

4. In statistics, the function e^{-x^2} comes up as part of the "normal distribution."

(a) **Group Discussion:** Why is finding $\int_0^1 e^{-x^2}\,dx$ difficult?

We don't have an antiderivative formula for e^{-x^2}.
None of our integration techniques work for this integral.

(b) Find the Maclaurin Series for e^{-x^2}.

Hint: Substitution

Start with the Maclaurin series for e^x: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Replace x with $-x^2$: $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$

(c) Find an *approximation* for $\int_0^1 e^{-x^2}\,dx$.

Use a $\underbrace{\text{partial sum of the Maclaurin series for } e^{-x^2}}_{\text{that is, a Maclaurin polynomial.}}$.

For example, if we use the Maclaurin polynomial for e^{-x^2} with degree 6:

$$\int_0^1 e^{-x^2}dx = \int_0^1 \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!}\right) dx$$

$$= \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42}\right]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} \approx 0.7428\ldots$$

This is close to the exact value $\int_0^1 e^{-x^2}dx = 0.7468\ldots$.