# MATH 126 A/B — 5 Nov. 2025

**RECALL:** A vector encodes <u>change</u> in $x, y, z$ from one point to another

*Example:* $A = (1,2,3)$  $B = (8,10,2)$

Vector from A to B is $\vec{AB} = \langle 7, 8, -1 \rangle$

## DOT PRODUCT:

The dot product is a way of "multiplying" two vectors. The result is a <u>number.</u>

In $\mathbb{R}^2$: $\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$

In $\mathbb{R}^3$: $\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$

In $\mathbb{R}^n$: $\langle a_1, a_2, \dots, a_n \rangle \cdot \langle b_1, b_2, \dots, b_n \rangle = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$

## The dot product tells us:

(a) Vectors $\vec{u}$ and $\vec{v}$ are orthogonal (perpendicular) exactly when $\vec{u} \cdot \vec{v} = 0$.

(b) $\vec{u} \cdot \vec{u} = |\vec{u}|^2$

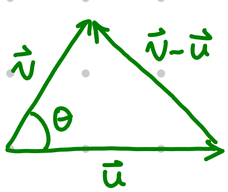
## Algebra and dot products:

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

$$(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$$

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

## Geometry and dot products:




If $\theta$ is the angle between $\vec{u}$ and $\vec{v}$, then:

$$|\vec{v} - \vec{u}|^2 = |\vec{u}|^2 + |v|^2 - 2|u||v| \cos \theta$$ ← Law of Cosines

$$(\vec{v} - \vec{u}) \cdot (\vec{v} - \vec{u}) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2|u||v| \cos \theta$$

$$-2 \vec{u} \cdot \vec{v} = -2|u||v| \cos \theta$$

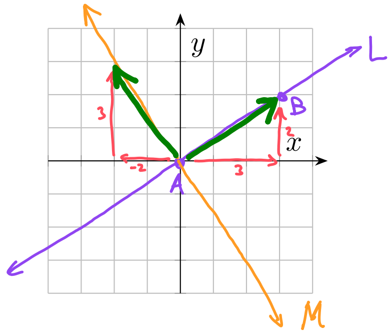
$$\boxed{\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta}$$ ← this is an important formula

# Dot Product

**1.** (a) In $\mathbb{R}^2$, draw the line $L$ through the points $A = (0,0)$ and $B = (3,2)$. What is the slope of $L$?

Perpendicular Lines:
slopes are negative reciprocals of each other

Perpendicular Vectors:
$\langle a,b \rangle$ is perpendicular to $\langle -b,a \rangle$
$\langle a,b \rangle \cdot \langle -b,a \rangle = 0$




Slope of L: $\frac{2}{3}$

slopes of perpendicular lines: flip the fraction and negate

(b) Draw the line $M$ through point $A$ that is *perpendicular* to $L$. What is the slope of line $M$?

slope of M: $-\frac{3}{2}$

(c) Find a vector $\mathbb{R}^2$ that is *parallel* to $L$. How is your vector related to the slope of $L$?

☞ Maybe use $A$ and $B$?

$\overrightarrow{AB} = \langle 3,2 \rangle$

other answers are possible, such as $\langle 6,4 \rangle$ or $\langle -3,-2 \rangle$

(d) Now, try to find a vector that is *parallel* to the line $M$ (and thus perpendicular to $L$). How is this vector related to the previous vector that you found?

$\vec{v} = \langle -2, 3 \rangle$ or any multiple of this, such as $\langle 1, -\frac{3}{2} \rangle$

Swap the x- and y-components, then negate one of them

**2.** (a) What is $\langle 3,1 \rangle \cdot \langle 1,2 \rangle$?

$\langle 3,1 \rangle \cdot \langle 1,2 \rangle = 3(1) + 1(2) = 3+2 = 5$

(b) What is $\langle -2,1,0 \rangle \cdot \langle 1,0,1 \rangle$?

$\langle -2,1,0 \rangle \cdot \langle 1,0,1 \rangle = -2(1) + 1(0) + 0(1) = -2$

(c) What is $\langle 3,2 \rangle \cdot \langle 2,-3 \rangle$?

☞ Have you seen this recently?

perpendicular vectors

$\langle 3,2 \rangle \cdot \langle 2,-3 \rangle = 3(2) + 2(-3) = 0$

(d) What is $\langle a,b,c \rangle \cdot \langle a,b,c \rangle$? How is this related to $|\langle a,b,c \rangle|$?

$\langle a,b,c \rangle \cdot \langle a,b,c \rangle = a^2 + b^2 + c^2 = |\langle a,b,c \rangle|^2$

For any vector $\vec{u}$: $\vec{u} \cdot \vec{u} = |\vec{u}|^2$

☞ Remember, $|\langle a,b,c \rangle|$ is the length of $\langle a,b,c \rangle$.

**3. Group chat:** Make a conjecture by filling in the blank.

Vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular (or *orthogonal*) exactly when $\mathbf{u} \cdot \mathbf{v}$ equals zero.

This is true for vectors in $\mathbb{R}^n$.

**4. Group experiment**: If $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors, how does $\mathbf{u} \cdot \mathbf{v}$ relate to $|\mathbf{u}|$ and $|\mathbf{v}|$?

☞ Choose some parallel vectors and see what happens!

example: $\vec{u} = \langle 1,5 \rangle$ $\quad |\vec{u}| = \sqrt{1^2+5^2} = \sqrt{26}$

$\vec{v} = \langle -2,-10 \rangle$ $\quad |\vec{v}| = \sqrt{(-2)^2+(-10)^2} = \sqrt{104} = 2\sqrt{26}$

$(\sqrt{26})(2\sqrt{26}) = 2(26) = 52$

$\vec{u} \cdot \vec{v} = \langle 1,5 \rangle \cdot \langle -2,-10 \rangle = 1(-2) + 5(-10) = -52$

For parallel vectors, $|\vec{u} \cdot \vec{v}| = |\vec{u}||\vec{v}|$

**5. Group chat:** Suppose **u**, **v**, and **w** are vectors of the same dimension, and $c$ is a number. Which of the following statements are (always) true? Which statements are false?

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ TRUE!

$\vec{u} \cdot \vec{v} = \langle u_1, u_2 \rangle \cdot \langle v_1, v_2 \rangle = u_1 v_1 + u_2 v_2$

$\vec{v} \cdot \vec{u} = \langle v_1, v_2 \rangle \cdot \langle u_1, u_2 \rangle = v_1 u_1 + v_2 u_2$

These are equal!

(b) $\underbrace{(\mathbf{u} \cdot \mathbf{v})}_{\text{number}} \cdot \mathbf{w} = \mathbf{u} \cdot \underbrace{(\mathbf{v} \cdot \mathbf{w})}_{\text{number}}$ FALSE

You can't do a dot product of a number and a vector.

You can't do the dot product of three vectors.

(c) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ TRUE!

$\vec{u} \cdot (\vec{v} + \vec{w}) = \langle u_1, u_2 \rangle \cdot (\langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle) = \langle u_1, u_2 \rangle \cdot \langle v_1 + w_1, v_2 + w_2 \rangle = u_1 v_1 + u_1 w_1 + u_2 v_2 + u_2 w_2$

$\vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} = \langle u_1, u_2 \rangle \cdot \langle v_1, v_2 \rangle + \langle u_1, u_2 \rangle \cdot \langle w_1, w_2 \rangle = u_1 v_1 + u_2 v_2 + u_1 w_1 + u_2 w_2$

these are equal

(d) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{v} \cdot \mathbf{u})$ TRUE!

$(c\vec{u}) \cdot \vec{v} = (c\langle u_1, u_2 \rangle) \cdot \langle v_1, v_2 \rangle = \langle cu_1, cu_2 \rangle \cdot \langle v_1, v_2 \rangle = cu_1 v_1 + cu_2 v_2$

$c(\vec{v} \cdot \vec{u}) = c(\langle v_1, v_2 \rangle \cdot \langle u_1, u_2 \rangle) = c(v_1 u_1 + v_2 u_2) = cv_1 u_1 + cv_2 u_2$

these are equal

**6.** Let $\mathbf{u} = \langle 2, 0, 4 \rangle$ and $\mathbf{v} = \langle -1, 2, 3 \rangle$. If $\theta$ is the angle between **u** and **v**, find $\cos\theta$.

$|\vec{u}| = \sqrt{2^2 + 0^2 + 4^2} = \sqrt{20}$ and $|\vec{v}| = \sqrt{(-1)^2 + 2^2 + 3^2} = \sqrt{14}$

$\vec{u} \cdot \vec{v} = -2 + 0 + 12 = 10$

Then $\cos(\theta) = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \dfrac{10}{\sqrt{20}\sqrt{14}} = \sqrt{\dfrac{5}{14}} \approx 0.598.$

So $\theta = \arccos\left(\sqrt{\frac{5}{14}}\right) = 0.93$ radians $= 53.3$ degrees

**7.** Find the angle between the vectors $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{v} = -2\mathbf{i} + 4\mathbf{j}$.

$|\vec{u}| = \sqrt{3^2 + 2^2} = \sqrt{13}$ and $|\vec{v}| = \sqrt{(-2)^2 + (4)^2} = \sqrt{20}$

$\vec{u} \cdot \vec{v} = -6 + 8 = 2$

Then $\cos(\theta) = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \dfrac{2}{\sqrt{13}\sqrt{20}} = \dfrac{1}{\sqrt{65}} \approx 0.124$

So $\theta = \arccos\left(\frac{1}{\sqrt{65}}\right) = 1.44$ radians $= 82.9$ degrees

**8.** Find the angle between the vectors $\langle 1, 0, 1 \rangle$ and **i**.

$|\vec{u}| = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $|\vec{v}| = 1$ and $\vec{u} \cdot \vec{v} = 1.$

Then $\cos(\theta) = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \dfrac{1}{\sqrt{2} \cdot 1} = \dfrac{1}{\sqrt{2}}$

So $\theta = \arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ radians $= 45$ degrees