MATH 126 A/B — 7 November 2025

CROSS PRODUCT: Only defined for vectors in $\mathbb{R}^3$.

If $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ then

$$\vec{u} \times \vec{v} = \langle u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1 \rangle.$$

Remember it like this:

$$\vec{u} \times \vec{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k}$$

where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ is the <u>determinant</u> of a 2×2 matrix

Example: $\vec{u} = \langle 1, 0, 2 \rangle$ and $\vec{v} = \langle 3, -1, 1 \rangle$

$$\vec{u} \times \vec{v} = \langle 0(1) - (2)(-1),\ (2)(3) - (1)(1),\ 1(-1) - 3(0) \rangle = \langle 2, 5, -1 \rangle$$

KEY PROPERTY: $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

OTHER PROPERTIES:

$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

$(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

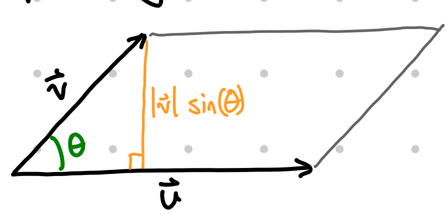
$(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$

$\vec{u} \times \vec{v} = \vec{0}$ if $\vec{u}$ and $\vec{v}$ are parallel

$|\vec{u} \times \vec{v}|$ is the area of parallelogram determined by $\vec{u}$ and $\vec{v}$.

<u>Note</u>: cross product is not associative

$$(\vec{u} \times \vec{v}) \times \vec{w} \neq \vec{u} \times (\vec{v} \times \vec{w})$$



Cross Product

1. Compute the following cross products

(a) $\langle 1,0,0\rangle \times \langle 0,1,0\rangle = \langle 0(0)-0(1),\ 0(0)-1(0),\ 1(1)-0(0)\rangle$

$= \langle 0,0,1\rangle$

$\vec{i} \times \vec{j} = \vec{k}$

(b) $\langle a,0,0\rangle \times \langle 0,b,0\rangle = \langle 0(0)-0(b),\ 0(0)-a(0),\ a(b)-0(0)\rangle$

$= \langle 0,0,ab\rangle$

(c) $\langle 1,1,1\rangle \times \langle 2,3,4\rangle = \langle 1(4)-1(3),\ 1(2)-1(4),\ 1(3)-1(2)\rangle$

$= \langle 1,-2,1\rangle$

2. If you did #1(c) correctly, you should have gotten $\langle 1,-2,1\rangle$.

Now compute both $\langle 1,1,1\rangle \cdot \langle 1,-2,1\rangle$ *and* $\langle 2,3,4\rangle \cdot \langle 1,-2,1\rangle$. What do you observe?

$= 1(1)+1(-2)+1(1)$

$= 0$

$= 2(1)+3(-2)+4(1)$

$= 2-6+4 = 0$

3. Group chat: Examine all of the cross products you computed in #1. Make a conjecture about how $\mathbf{a} \times \mathbf{b}$ is related to vectors $\mathbf{a}$ and $\mathbf{b}$.

$\vec{a} \times \vec{b}$ is always perpendicular to $\vec{a}$ and to $\vec{b}$

4. What is $\langle 2,3,4\rangle \times \langle 1,1,1\rangle$?

$= \langle 3(1)-4(1),\ 4(1)-2(1),\ 2(1)-3(1)\rangle = \langle -1,2,-1\rangle$

5. Group chat: What do you think is the relationship between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$?

$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

6. Experiment time: If $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors, what can you say about $\mathbf{b} \times \mathbf{a}$? or $\vec{a} \times \vec{b}$?

EXAMPLE: $\vec{a} = \langle 2,4,6\rangle$, $\vec{b} = \langle \frac{2}{3}, \frac{4}{3}, \frac{6}{3}\rangle$

$\vec{a} \times \vec{b} = \langle 4(\frac{6}{3})-6(\frac{4}{3}),\ 6(\frac{2}{3})-2(\frac{6}{3}),\ 2(\frac{4}{3})-4(\frac{2}{3})\rangle = \langle 0,0,0\rangle$

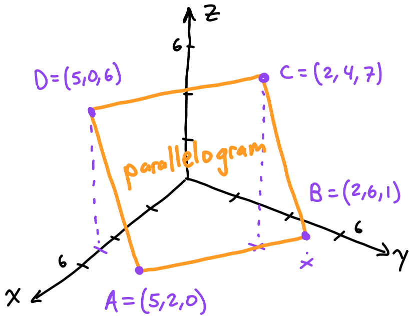
This is the zero vector, denoted $\vec{0}$.

If $\vec{a}$ and $\vec{b}$ are parallel, then $\vec{a} \times \vec{b} = \vec{0}$.

7. Consider the points $A = (5, 2, 0)$, $B = (2, 6, 1)$, $C = (2, 4, 7)$, and $D = (5, 0, 6)$.

(a) Sketch these points in $\mathbb{R}^3$ and show that they form a parallelogram.

(b) Use a cross product to find the area of this parallelogram.



$\overrightarrow{AB} = \langle 2-5, 6-2, 1-0 \rangle = \langle -3, 4, 1 \rangle$

$\overrightarrow{DC} = \langle 2-5, 4-0, 7-6 \rangle = \langle -3, 4, 1 \rangle$

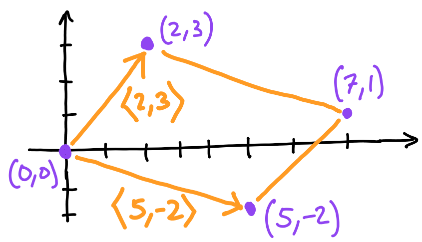
$\overrightarrow{AD} = \overrightarrow{BC} = \langle 0, -2, 6 \rangle$

Area of parallelogram:

$|\overrightarrow{AB} \times \overrightarrow{DC}| = |\langle -3,4,1 \rangle \times \langle 0,-2,6 \rangle| = |\langle 26, 18, 6 \rangle| = \sqrt{1036} \approx 32.187$

8. Find the area of the parallelogram with vertices $(0, 0)$, $(5, -2)$, $(7, 1)$, and $(2, 3)$.

☜ These points are in $\mathbb{R}^2$. Is that a problem?



View these vectors as in $\mathbb{R}^3$ with $z = 0$.

Area of parallelogram:

$$|\langle 5, -2, 0 \rangle \times \langle 2, 3, 0 \rangle| = |\langle 0, 0, 15+4 \rangle| = 19$$

So the area of the parallelogram is 19.

Bonus: we'll come back to this next week!

9. Cleo: Hey Bastian, remember how a line is determined by two points?

Bastian: I sure do! Did you know that in $\mathbb{R}^3$ a plane is determined by three points?

Cleo: Yes—as long as all three points are not all on the same line.

Group chat: Does this make sense?

Challenge question: The plane P passes through the points $A = (-1, 2, 3)$, $B = (0, 3, 4)$, and $C = (1, 5, 7)$. Can you find a vector $\mathbf{n}$ that is orthogonal (perpendicular) to the plane P?