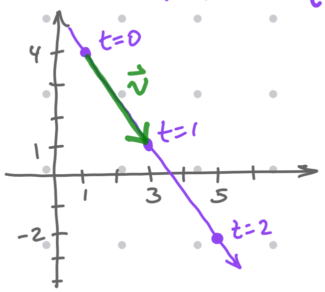
MATH 126 A/B — 10 November 2025

PROBLEM: A car is driving around $\mathbb{R}^2$.
At $t=0$, the car is at $(1,4)$.
In t seconds, the car moves $2t$ units right and $3t$ units down.

Questions: What is the shape of motion?
What equation gives the car's position at time t?



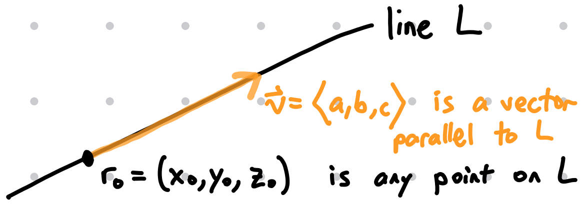
Parametric equations: $\begin{cases} x = 1 + 2t \\ y = 4 - 3t \end{cases}$

Vector $\vec{v} = \langle 2, -3 \rangle$ is the <u>direction</u> vector of the car's motion

Vector equation for the line: $\vec{r}(t) = \langle 1,4 \rangle + t\langle 2,-3 \rangle$.

$\vec{r}(t)$: position $\langle x,y \rangle$ at time t; $\langle 1,4 \rangle$: initial position; $\langle 2,-3 \rangle$: direction vector

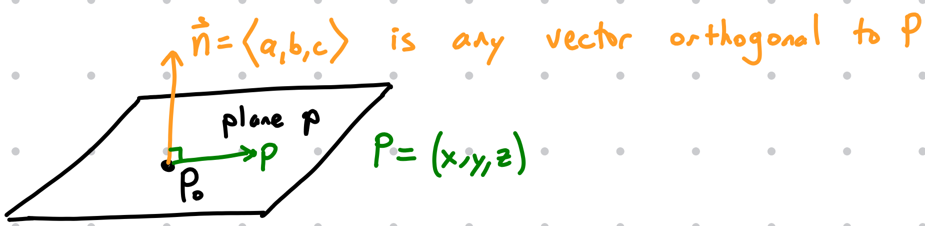
In ANY dimension: A point and a direction vector determine a line.



Vector equation: $\vec{r}(t) = \vec{r}_0 + t\vec{v}$

Parametric eqs: $\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$

GENERAL STRATEGY FOR THE EQUATION OF A PLANE:



$P_0 = (x_0, y_0, z_0)$, a known point on p

An equation for plane p: $\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

↓ rearrange

$$ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}_{\text{a number}}$$

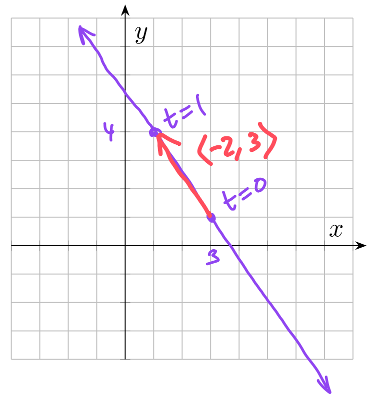
Lines and Planes

1. Group investigation: Pick several different values of t and plot the points whose x and y coordinates are given by:

☞ Everyone at your table can pick different values of t. Then share your answers.

parametric equations

$$\begin{cases} x = 3 - 2t \\ y = 1 + 3t \end{cases}$$



Vector equation of the line:

$\langle x, y \rangle = \langle 3, 1 \rangle + t\langle -2, 3 \rangle$

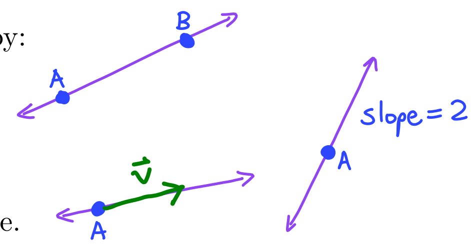
What do you notice?

2. True or False: In two dimensions, a line is determined by:

TRUE (a) Two different points on the line.

TRUE (b) One point on the line and the slope of the line.

TRUE (c) One point on the line and a vector parallel to the line.



3. True or False: In *three* dimensions, a line is determined by:

TRUE (a) Two different points on the line.

FALSE (b) One point on the line and the slope of the line.

Slope isn't enough in 3D!

TRUE (c) One point on the line and a vector parallel to the line.

4. Write equations for the line passing through the point $(4, 7, -2)$ and parallel to the vector $\langle 3, -1, 2 \rangle$...

Let $P = (4, 7, -2)$

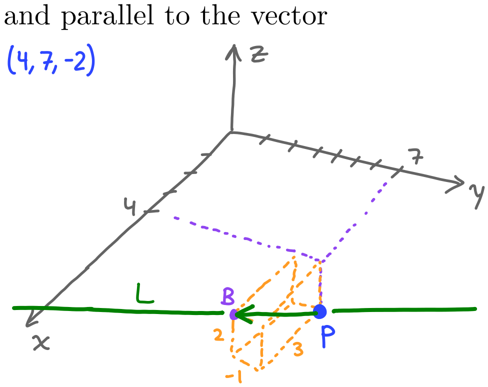
Let $\vec{v} = \langle 3, -1, 2 \rangle$

(a) In vector form:

$$\underbrace{\langle x, y, z \rangle}_{} = \underbrace{\langle 4, 7, -2 \rangle}_{P} + \underbrace{t}_{+\ t} \underbrace{\langle 3, -1, 2 \rangle}_{\vec{v}}$$

(b) In parametric form:

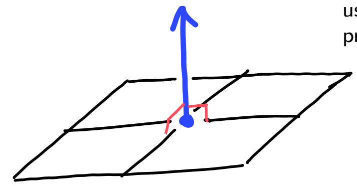
$$\begin{cases} x = 4 + 3t \\ y = 7 - t \\ z = -2 + 2t \end{cases}$$



$B = P + \vec{v} = \langle 4, 7, -2 \rangle + \langle 3, -1, 2 \rangle = \langle 7, 6, 0 \rangle$

☞ Make drawings, use your hands, use props, etc.!

5. True or False: In three dimensions, a *plane* is determined by:



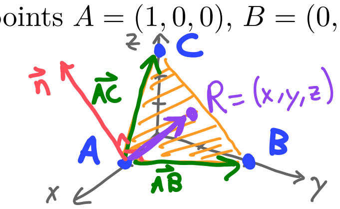
TRUE (a) Three points (not all in the same line)

TRUE (b) One point and two different (non-parallel) direction vectors

TRUE (c) $\underbrace{\text{One point}}_{\text{in the plane}}$ and $\underbrace{\text{a perpendicular vector}}_{\text{to the plane}}$

6. Let's think about the plane P that goes through the points $A = (1, 0, 0)$, $B = (0, 2, 0)$, and $C = (0, 0, 3)$.

(a) Make a sketch of the plane P.



(b) One vector parallel to the plane is $\overrightarrow{AB} = \langle -1, 2, 0\rangle$. A different vector parallel to P is $\overrightarrow{AC} = \langle -1, 0, 3\rangle$. How can you find a vector $\mathbf{n}$ that is *perpendicular* to the plane P?

☞ Hint: You have TWO vectors that are parallel to the plane.

normal vector: $\vec{n} = \vec{AB} \times \vec{AC} = \langle -1, 2, 0\rangle \times \langle -1, 0, 3\rangle = \langle 6, 3, 2\rangle$

(c) Suppose that $R = (x, y, z)$ is a random point located in plane P. Explain why the vector $\overrightarrow{AR} = \langle x - 1, y - 0, z - 0\rangle$ is parallel to the plane P.

Since A and R are points in the plane, vector $\vec{AR}$ is parallel to the plane.

(d) We are going to find $\mathbf{n} \cdot \overrightarrow{AR}$ in two ways:

- Compute $\mathbf{n} \cdot \overrightarrow{AR}$ using the definition of the dot product.
- Since $\overrightarrow{AR}$ is parallel to P and $\mathbf{n}$ is perpendicular to P, what is $\mathbf{n} \cdot \overrightarrow{AR}$?

$\vec{n} \cdot \vec{AR} = \langle 6, 3, 2\rangle \cdot \langle x-1, y-0, z-0\rangle$ and also $\vec{n} \cdot \vec{AR} = 0$

(e) **Milo:** "WOW! That means an *equation* for the plane P is $6(x - 1) + 3(y - 0) + 2(z - 0) = 0$."
Group chat: How did Milo arrive at this equation?

Thus: $6(x-1) + 3(y-0) + 2(z-0) = 0$

or: $\boxed{6x + 3y + 2z = 6}$

(f) Explain why $6(x - 0) + 3(y - 2) + 2(z - 0) = 0$ is another way to write an equation for plane P.

coordinates of point B, so $\vec{n} \cdot \vec{BR} = 0$

You can use any point in the plane to find an equation for the plane.

(g) What is the *general strategy* for finding the equation of a plane?

Find a normal vector $\vec{n}$, which is orthogonal to the plane.
Find a point A in the plane. Let R=(x,y,z) be a random point in the plane
Then $\vec{n} \cdot \vec{AR} = 0$ gives an equation of the plane.

7. Suppose that P is the plane that passes through the points $A = (-1, 2, 3)$, $B = (0, 4, 4)$, and $C = (-1, 2, 4)$. Find an equation for P.

Vectors parallel to plane P: $\vec{AB} = \langle 1, 2, 1\rangle$ and $\vec{AC} = \langle 0, 1, 1\rangle$.

Normal vector to the plane: $\vec{n} = \vec{AB} \times \vec{AC} = \langle 1,2,1\rangle \times \langle 0,1,1\rangle = \langle 1, -1, 1\rangle$.

Equation of the plane: $\vec{n} \cdot \left(A - (x,y,z)\right) = \langle 1,-1,1\rangle \cdot \langle x-(-1), y-2, z-3\rangle = 0$

$1(x+1) - 1(y-2) + 1(z-3) = 0$

$x + 1 - y + 2 + z - 3 = 0$ $\longrightarrow$ $\boxed{x - y + z = 0}$

8. *Spicy*: Find the angle between the planes $x + y + z = 1$ and $x - y + 3z = 3$.

Hint: the angle between two planes is the same as the angle between the normal vectors of the two planes.