Directional Derivatives

1. The wind chill function $w(T, v)$ is on the screen again. As we did last week, focus on the entry corresponding to $T = 10$ and $v = 15$. That is, $w(10, 15) = -7$.

Recall from last week:

- If $\Delta T = 5$, then $\Delta W = 7$. So, an *estimate* for the partial derivative in the T direction is $w_T(15, 20) \approx \frac{7}{5}$.
- If $\Delta v = 5$, then $\Delta W = -2$. So, an *estimate* for the partial derivative in the v direction is $w_v(15, 20) \approx -\frac{2}{5}$.

(a) What is your best guess for the rate that w changes if $\Delta T = 5$ *and* $\Delta v = 5$?

☞ Both variables change!

(b) What is your best guess for the rate that w changes if $\Delta T = 1$ *and* $\Delta v = 1$?

☞ Both variables change!

(c) **Milo:** Hey Jade, $w_T(10, 15)$ is really a rate of change!
Jade: It sure is, a rate of change, Milo! That means we can think about it like this:

$$w_T(10,15) \approx \frac{\Delta w}{\Delta T} = \frac{w(15,15) - w(10,15)}{5}$$
$$= \frac{w(15,15) - w(10,15)}{\text{the distance between the points } (T,v) = (15,15) \text{ and } (T,v) = (10,15)}$$

Group chat: Why is the previous fraction actually equal to $\frac{7}{5}$?

(d) **Milo:** Could we also talk about a rate of change in the direction where $\Delta T = 5$ *and* $\Delta v = 5$?
Jade: We sure can! In this case, our fraction becomes:

$$\text{rate of change} = \frac{w(15,20) - w(10,15)}{\text{the distance between the points } (T,v) = (15,20) \text{ and } (T,v) = (10,15)}$$

Group chat: What is this new fraction equal to?

✋ *Stop and wait for further instructions.* While you wait, discuss: Would it make sense to talk about the rate that w changes for other combinations of ΔT and Δv?

2. What is the rate at which $f(x, y) = 2x^2 + y^2 - 5$ is changing in the direction $\mathbf{u} = \langle 3, 4\rangle$ at the point $(x, y) = (1, 3)$?

3. **Milo:** WOW! We have a formula for the directional derivative when $\mathbf{u}$ is the unit vector $\langle a, b\rangle$:

$$D_{\mathbf{u}}f(x, y) = f_x(x, y) \cdot a + f_y(x, y) \cdot b.$$

Jade: Milo, look!! $D_{\mathbf{u}}f(x, y)$ is the *dot product* of two vectors!

Group chat: Which two vectors is $f_x(x, y) \cdot a + f_y(x, y) \cdot b$ the dot product of?

$$f_x(x, y) \cdot a + f_y(x, y) \cdot b = \Big\langle \qquad , \qquad \Big\rangle \bullet \Big\langle \qquad , \qquad \Big\rangle$$

Milo: One of those vectors is $\mathbf{u}$. The other vector must be important, so maybe we should give it a name?

4. **Jade:** Hey, do you remember the formula $\mathbf{v} \bullet \mathbf{u} = |\mathbf{v}|\,|\mathbf{u}|\,\cos(\theta)$?

Milo: I sure do! Let's try to apply it to $D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \bullet \mathbf{u}$.

Jade: OK, if we substitute the vectors into the formula, we get

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \bullet \mathbf{u} = |\nabla f(x, y)|\,|\mathbf{u}|\,\cos(\theta).$$

What good is that?

Milo: Well, we know $|\mathbf{u}| = 1$ and we know $|\nabla f(x, y)|$ is some number. So, $D_{\mathbf{u}}f(x, y)$ is as large as it could possibly be when $\cos(\theta)$ equals 1.

Group chat: What angle θ makes $D_{\mathbf{u}}f(x, y)$ as large as it could possibly be? What does this mean about the vectors $\nabla f(x, y)$ and $\mathbf{u}$?

Group chat: What angle θ makes $D_{\mathbf{u}}f(x, y)$ as *small** as it could possibly be? What does this mean about the vectors $\nabla f(x, y)$ and $\mathbf{u}$?

*Negative numbers are, in fact, smaller than 0.

5. Let $f(x, y) = xy + 2x^2 - 3y$. On the graph of f, what is the direction of the steepest slope at the point $(2, 1)$? What is this steepest slope?