MATH 126 A/B — 1 December 2025

Double Integrals

1. *Warm-up*: Standing outside of Buntrock Commons on a snowy day, suppose you measure the **rate** (in cm per hour) at which snow is falling every 30 minutes. Here is the data:

Time of Day	12:00	12:30	1:00	1:30	2:00	2:30	3:00
Rate of snowfall (cm per hour)	1	2	3	1	0	2	3

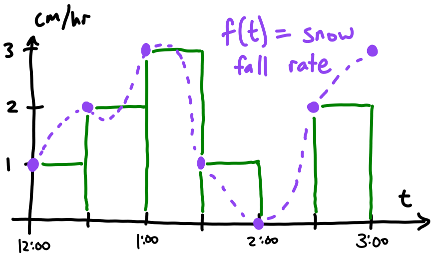
(a) **Group chat:** How can you estimate the actual number of centimeters of snow that fell between 12:00 and 12:30?

It appears that the rate of snowfall was between 1 and 2 cm per hour

If it snowed at 1.5 cm per hour, then 0.75 snow would fall in half an hour.

(b) **Group chat:** What is a good estimate for the actual number of centimeters of snow that fell between 12:00 and 3:00?

☞ There is more than one way to estimate this!



We could use a left Riemann sum:

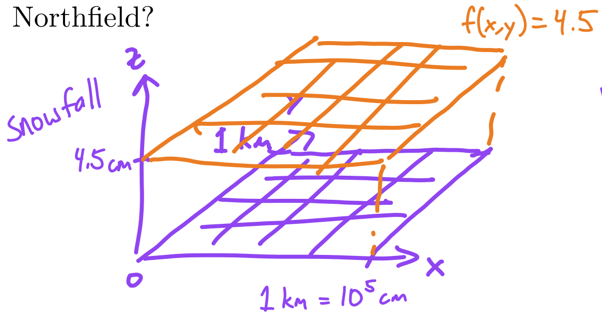
$$\text{total} = (1+2+3+1+0+2 \text{ cm/hr})(\tfrac{1}{2}\text{ hr})$$

$$= \frac{9}{2}\text{ cm} = 4.5\text{ cm}$$

2. Now suppose the snow fell at the *same* rate all throughout Northfield (that is, the rate does not change based on location). For simplicity, assume that Northfield is a one square kilometer region (that is, 100,000 × 100,000 cm).

Group chat: What is a good estimate for the actual volume of snow (in cm^3) collected in Northfield?

☞ You can measure volume in cm^3.



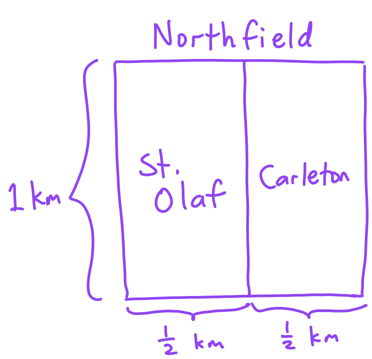
$$\text{Volume} = (4.5\text{ cm})(10^5\text{ cm})(10^5\text{ cm})$$

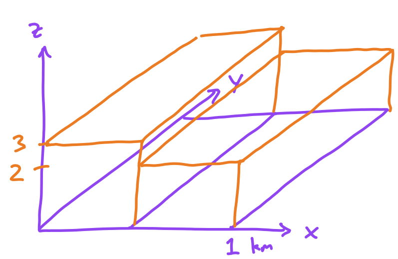
$$= 4.5 \times 10^{10}\text{ cm}^3$$

$$= 4.5 \times 10^4\text{ m}^3$$

3. Now suppose that snow is falling at a rate of 3 cm per hour on the St. Olaf half of Northfield and at a rate of 2 cm per hour on the Carleton half of Northfield.

Group chat: What is a good estimate for the volume of snowfall in Northfield in 1 hour?





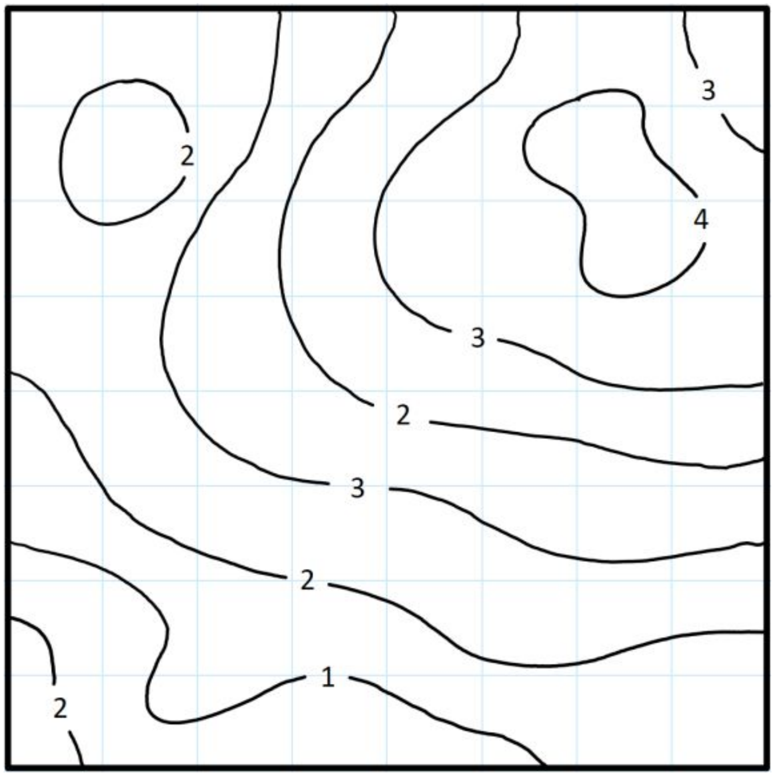
total volume:

$$(3\text{cm})(5\times10^4\text{cm})(10^5\text{ cm})$$

$$+ (2\text{cm})(5\times10^4\text{ cm})(10^5\text{ cm})$$

$$= 2.5 \times 10^{10}\text{ cm}^3$$

4. Now suppose the rate at which snow falls is *different* throughout Northfield. Here is a contour map of the different rates in a 1 km square region at 12:00:



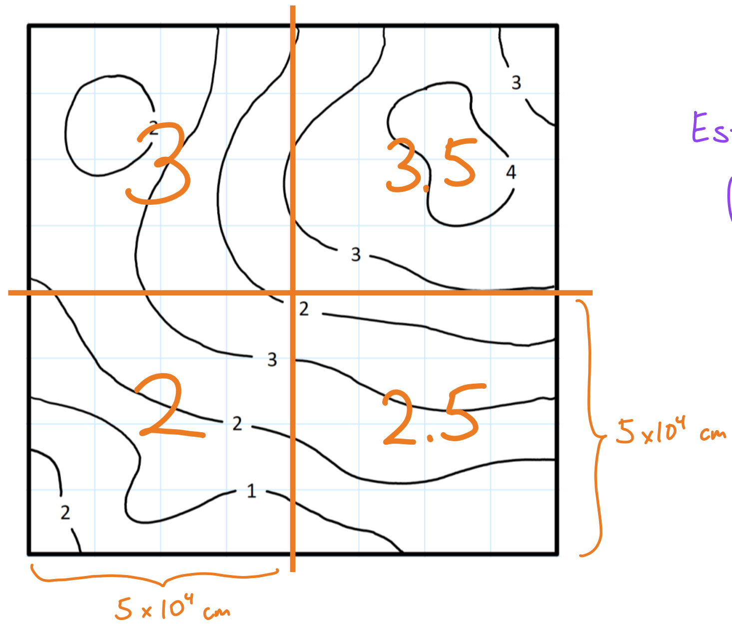
(a) **Group chat:** How can you estimate the volume of snow that fell within this region between 12:00 and 12:30?

IDEA: Split both the x- and y-directions into smaller pieces. This results in the xy-region being broken up into rectangles. Assume $f(x,y)$ is constant on each.

See the next page for some possible estimates

(b) **Group chat:** Whatever you did in part (a), what could you do to improve your estimate?

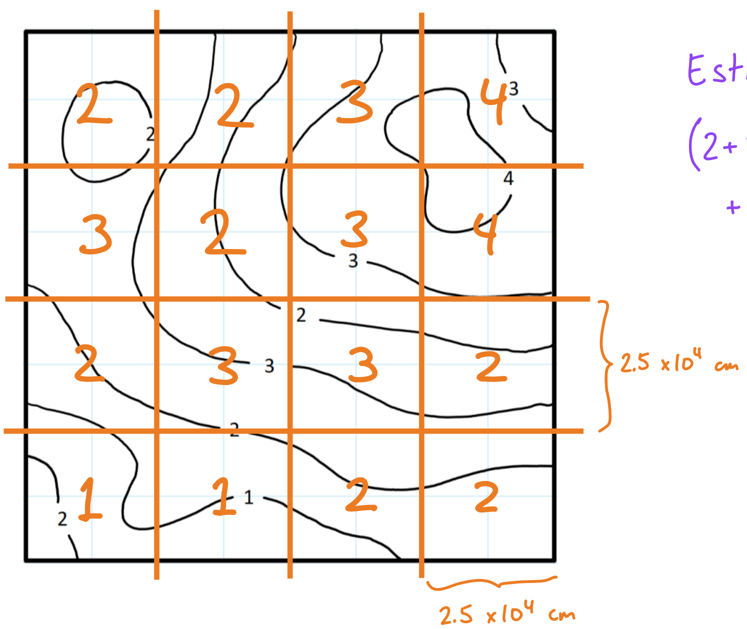
Use smaller rectangles!



Estimate:

$$\left(3+3.5+2+2.5\ \text{cm/hr}\right)\left(5\times10^{4}\ \text{cm}\right)\left(5\times10^{4}\ \text{cm}\right)\left(\tfrac{1}{2}\ \text{hr}\right)$$

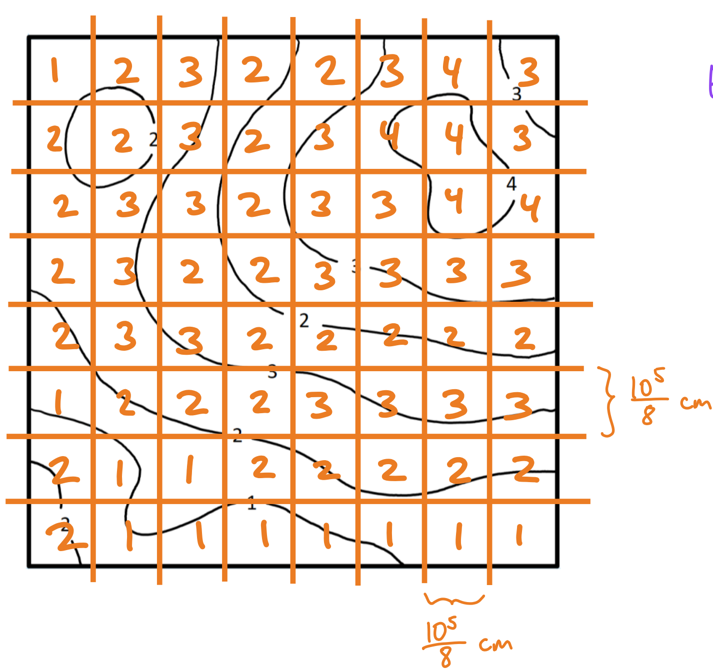
$$=1.38\times10^{10}\ \text{cm}^{3}$$



Estimate:

$$\left(2+2+3+4+3+2+3+4+2+3+3+2+1+1+2+2\ \text{cm/hr}\right)\left(2.5\times10^{4}\ \text{cm}\right)\left(2.5\times10^{4}\ \text{cm}\right)\left(\tfrac{1}{2}\ \text{hr}\right)$$

$$=1.22\times10^{10}\ \text{cm}^{3}$$



Estimate:

$$\left(\text{Sum of all 64 numbers at left (cm/hr)}\right)\left(\frac{10^{5}}{8}\ \text{cm}\right)\left(\frac{10^{5}}{8}\ \text{cm}\right)\left(\tfrac{1}{2}\ \text{hr}\right)$$

$$=\left(148\ \text{cm/hr}\right)\left(\frac{10^{10}}{128}\ \text{cm}^{2}\cdot\text{hr}\right)$$

$$=1.16\times10^{10}\ \text{cm}^{3}$$

5. Sketch the solid under the graph of $f(x, y) = 5$ and above the region $R = \underbrace{[0, 3]}_{x} \times \underbrace{[0, 2]}_{y}$. Then evaluate the following integral.

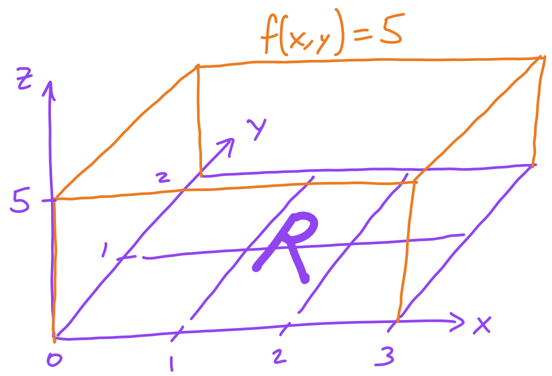
☞ R is the set of points (x, y) such that $0 \leq x \leq 3$ and $0 \leq y \leq 2$.

function to integrate

DOUBLE INTEGRAL: $$\iint_R 5\, dA$$

region in the xy-plane

integral with respect to area

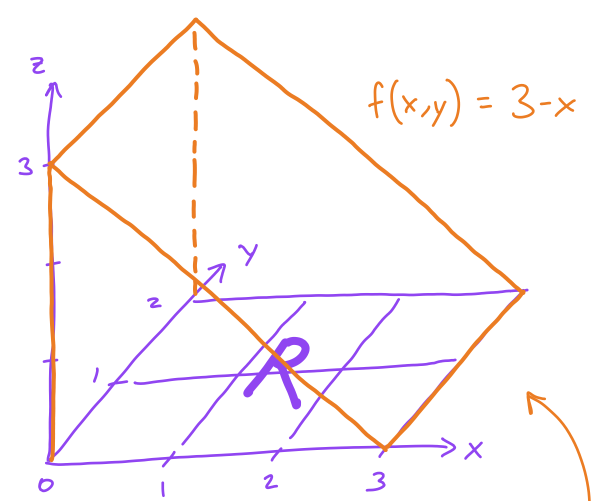


$$\iint_R 5\ dA = \text{volume under } f(x,y) \text{ above rectangle } R$$

$$= 3(2)(5) = 30$$

6. Sketch the solid under the graph of $f(x, y) = 3 - x$ and above the region $R = [0, 3] \times [0, 2]$. Then evaluate the following integral.

$$\iint_R (3 - x)\, dA$$



$$\iint_R (3-x)\ dA = \text{volume under } f(x,y) \text{ above region } R$$

$$= \tfrac{1}{2}(3)(3)(2)$$

$$= 9$$

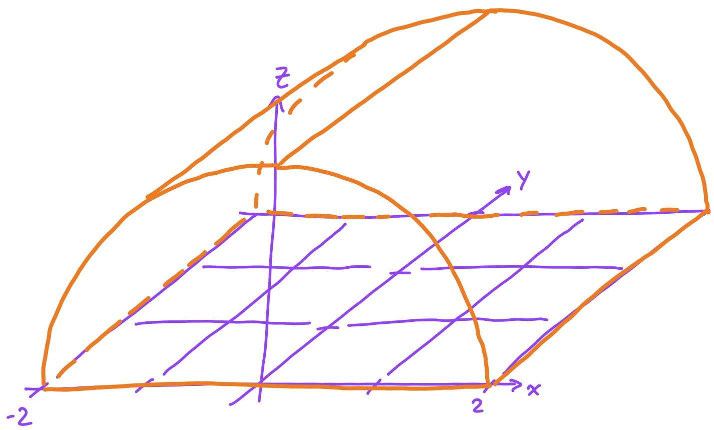
The volume of this "wedge" is half the volume of a box with dimensions 3×3×2.

We didn't do this problem in class:

7. Sketch the solid under the graph of $f(x, y) = \sqrt{4 - x^2}$ and above the region $R = [-2, 2] \times [0, 3]$. Then evaluate the following integral.

$$\iint\limits_R \sqrt{4 - x^2}\, dA$$

$f(x,y) = \sqrt{4-x^2}$



$\iint\limits_R \sqrt{4-x^2}\, dA =$ volume of half cylinder with radius 2 and height 3

$= \frac{1}{2}(\pi \cdot 2^2)(3) = 6\pi$

We will come back to this problem on Wednesday.

8. Sketch the solid between the graph of $f(x, y) = 1 - y$ and the region $R = [0, 2] \times [-2, 2]$. Then evaluate the following integral.

Why does this problem say "between" instead of "under"?

$$\iint\limits_R (1 - y)\, dA$$