Math 234

Equivalence Relations Day 23

1. Let $A = \{10, 11, 12, 13, 14\}$. The relation

$$\mathbf{R} = \{(10, 10), (10, 14), (11, 11), (11, 13), (12, 12), (13, 11), (13, 13), (14, 10), (14, 14)\}$$

is an equivalence relation. What are the equivalence classes of **R**?

2. Let **E** be a relation on the set **Z** of all integers defined by

$$m \mathbf{E} n \Leftrightarrow 4 \mid (m - n).$$

(a) Prove that this relation **E** is an equivalence relation by showing it is reflexive, symmetric and transitive.

(b) Describe the equivalence class $[0]$ of **E**.

(c) Describe the equivalence class $[1]$ of **E**.

(d) Describe the equivalence class $[2]$ of **E**.

(e) Describe the equivalence class $[-31]$ of **E**.

(f) Describe all the equivalence classes of **E**.

3. Let $A = \mathbf{Z} \times \mathbf{Z}$. Define a relation R on A as follows: For all (a, b) and (c, d) in A,

$$(a, b) \ \mathsf{R} \ (c, d) \Leftrightarrow a + d = c + b.$$

(a) Is it true that $(1, 2)$ R $(3, 4)$? How about $(-1, 4)$ R $(0, 5)$?

(b) Is R reflexive?

(c) Is R symmetric?

(d) Is R transitive?

(e) Is R an equivalence relation?

(f) List four elements of $[(1, 3)]$.

(g) List four elements of $[(-2, 6)]$.

(h) Describe all the equivalence classes of R.

4. Let X be a finite set. For all sets $U \in \mathscr{P}(X)$, let $N(U)$ denote the number of elements in U. Define a relation R on $\mathscr{P}(X)$ by U R V if and only if $N(U) = N(U)$.

Show that R is an equivalence relation. What are the equivalence classes of R?

5. Which of the following are partitions of the set $\mathbf{Z} \times \mathbf{Z}$ of ordered pairs of integers?

   (a) the set of pairs (x, y) where x or y is odd, the set of pairs (x, y) where x is even, and the set of pairs (x, y) where y is even

   (b) the set of pairs (x, y) where both x and y is odd, the set of pairs (x, y) where exactly one of x and y is odd, and the set of pairs (x, y) where both x and y are even

   (c) the set of pairs (x, y) where $3 \mid x$ and $3 \mid y$, the set of pairs (x, y) where $3 \mid x$ and $3 \nmid y$, the set of pairs (x, y) where $3 \nmid x$ and $3 \mid y$, the set of pairs (x, y) where $3 \nmid x$ and $3 \nmid y$

   ☞ the symbol $\nmid$ means "does not divide"

6. A partition P_1 is called a **refinement** of a partition P_2 if every set in P_1 is a subset of some set in P_2.

   (a) Show that the partition formed from congruence classes modulo 6 is a refinement of the partition formed from congruence classes modulo 3.

   (b) If the partition formed from congruence classes modulo p is a refinement of the partition formed from congruence classes modulo q, what can you say about p and q?