Math 234

Modular Arithmetic and $\mathbf{Z}_n$ Day 24

1. (a) What is the value of $23 \pmod 7$?

   (b) What is the value of $-12 \pmod 7$?

   (c) Is it true that $23 \pmod 7 = -12 \pmod 7$?

   (d) Is it true that $23 \equiv 12 \pmod 7$?

2. Using the facts that $46 \equiv 7 \pmod{13}$ and $17 \equiv 4 \pmod{13}$, using modular arithmetic to efficiently find an integer $0 \le d \le 12$ such that

   (a) $63 \equiv d \pmod{13}$. Note that $63 = 46 + 17$.

   (b) $29 \equiv d \pmod{13}$. Note that $29 = 46 - 17$.

   (c) $782 \equiv d \pmod{13}$. Note that $782 = 46 \times 17$.

   (d) $143 \equiv d \pmod{13}$. Note that $143 = (2 \times 46) + (3 \times 17)$.

3. Find the units digit of 7^{2022}. Then do the same for 37^{2022}. mod 10!

4. Recall that a **binary operation** on a set S is a function from $S \times S$ to S. Determine whether each of the functions below is a binary operation, and if so, identify set set S.

   (a) The logical *or* operation, as in $r \vee s$, where r and s are logical true/false values.

   (b) The logical *and* operation, as in $r \wedge s$, where r and s are logical true/false values.

   (c) The logical implication operation, as in $r \rightarrow s$, where r and s are logical true/false values.

   (d) The numerical less than operation, as in $r < s$, where r and s are real numbers.

5. Let $\cdot$ be the usual multiplication operation for real numbers in some set S.

   (a) If $S = \mathbf{R}$, is $\cdot$ a binary operation?

   (b) If $S = \mathbf{R}^+$, is $\cdot$ a binary operation?

   (c) If $S = \mathbf{Z}$, is $\cdot$ a binary operation?

   (d) If $S = \mathbf{Z}^-$ (negative integers), is $\cdot$ a binary operation?

6. Let $+$ be the usual addition operation on real numbers.

   (a) If $A = \{x \mid x > 0\}$, is A closed under $+$?

   (b) If $A = 2\mathbf{Z}$, the set of even integers, is A closed under $+$?

   (c) If $A = \{n \in \mathbf{Z} \mid n \text{ is odd}\}$, is A closed under $+$?

   (d) If $A = \mathbf{Q}$, is A closed under $+$?

   (e) If $A = \mathbf{R} - \mathbf{Q}$, is A closed under $+$?

7. Let $S = \{q + r\sqrt{2} \mid q, r \in \mathbf{Q}\}$ with the usual addition and multiplication of real numbers. Complete the following to establish that S is a commutative ring.

(a) Show that $+$ and $\cdot$ are commutative.

(b) Show that $+$ and $\cdot$ are associative.

(c) Show that $+$ distributes over $\cdot$.

(d) Show that S contains an additive identity and a multiplicative identity.

(e) Show that each element of S has an additive inverse.

8. Let S be the set of all 2×2 matrices of real numbers. Let $+$ be the usual matrix addition from linear algebra, and define a new "componentwise" multiplication $\star$ as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \star \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae & bf \\ cg & dh \end{bmatrix}$$

(a) Are $+$ and $\star$ commutative?

(b) Are $+$ and $\star$ associative?

(c) Do $+$ and $\star$ satisfy the distributive property?

(d) Is there an additive identity and a multiplicative identity?

(e) Are there additive inverses?

(f) Is S with $+$ and $\star$ a commutative ring?

9. **BONUS:** Find the units digit of 42^{4017}.