Homework 21

Math 234

due at classtime on Thursday, December 1

Write your solutions to the following problems clearly and neatly. You may write or type your solutions electronically, or write them on paper and scan or photograph them. Upload a single file containing your solutions to the Homework 21 assignment on the Moodle page for Math 234.

1. Which of the following functions is asymptotically smaller than the other?

$$f(n) = n^{14} \qquad\qquad g(n) = 14^n$$

2. Let $f(n) \ll h(n)$ and $g(n) \ll h(n)$. Prove that $f(n) + g(n) \ll h(n)$.

3. Let $f(n) = n!$ and $g(n) = n \cdot n!$. Are these functions asymptotically similar? If not, which is asymptotically smaller? Use limits to show that your answer is correct.

4. Let $f(n) = 2^n + 3$ and $g(n) = n^4 + 23$.

- **(a)** Compute $f(1)$ and $g(1)$. Which is smaller?
- **(b)** Compute $f(2)$ and $g(2)$. Which is smaller?
- **(c)** Compute $f(3)$ and $g(3)$. Which is smaller?
- **(d)** Are $f(n)$ and $g(n)$ asymptotically similar? If not, which is asymptotically smaller?

5. Determine whether each of the following functions is $O(x)$. Explain your answers.

- **(a)** $f(x) = 10$
- **(b)** $f(x) = x^2 + x + 1$
- **(c)** $f(x) = 3x + 4$
- **(d)** $f(x) = \lfloor x \rfloor$

6. Determine whether each of the following functions is $O(x^2)$. Explain your answers.

- **(a)** $f(x) = x \log x$
- **(b)** $f(x) = x^2 + 1000$
- **(c)** $f(x) = 2^x$
- **(d)** $f(x) = \lfloor x \rfloor \cdot \lceil x \rceil$

7. Show that x^3 is $O(x^4)$ but x^4 is not $O(x^3)$.

8. Use big-O notation to express the asymptotic growth rate of each of the following functions. In each case, when you say that $f(n)$ is $O(g)$, your function g should have the smallest possible growth rate.

- **(a)** $f(n) = (n^2 + 3)(n - 2)$
- **(b)** $f(n) = (2 \log n)(n + 5)$
- **(c)** $f(n) = 2^n(n^3 + 7n - 1)$
- **(d)** $f(n) = (n + 2^n)(n! + 99)$

9. Recall from linear algebra the algorithm for multiplying two $n \times n$ matrices.

If $n = 1$, the algorithm requires 0 additions and 1 multiplication.

$$[a] \cdot [b] = [ab]$$

If $n = 2$, the algorithm requires $1 \cdot 2^2 = 4$ additions and $2^3 = 8$ multiplications.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + bg & cf + dh \end{bmatrix}$$

If $n = 3$, the algorithm requires $2 \cdot 3^2 = 18$ additions and $3^3 = 27$ multiplications.

If $n = 4$, the algorithm requires $3 \cdot 4^2 = 48$ additions and $4^3 = 64$ multiplications.

(a) How many additions are required to multiply two $n \times n$ square matrices? Your answer should depend on n.

(b) How many multiplications are required to multiply two $n \times n$ square matrices? Your answer should depend on n.

(c) Using big-O notation, how many total arithmetic operations (additions plus multiplications) are needed to multiply two $n \times n$ matrices?

(d) If multiplying two 1000×1000 matrices together takes 5 seconds of CPU time, how much time, in minutes, should we expect the multiplication of two $10{,}000 \times 10{,}000$ matrices to take? How much time for two $500{,}000 \times 500{,}000$ matrices?