CASSINI'S IDENTITY: $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$

proof:

$$\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix}=\begin{bmatrix}2&1\\1&1\end{bmatrix}$$

$$\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix}=\begin{bmatrix}3&2\\2&1\end{bmatrix}$$

$$\begin{bmatrix}3&2\\2&1\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix}=\begin{bmatrix}5&3\\3&2\end{bmatrix}$$

$$\vdots$$

$$\begin{bmatrix}F_{n+1}&F_n\\F_n&F_{n-1}\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix}=\begin{bmatrix}F_{n+1}+F_n&F_{n+1}\\F_n+F_{n-1}&F_n\end{bmatrix}=\begin{bmatrix}F_{n+2}&F_{n+1}\\F_{n+1}&F_n\end{bmatrix}$$

for integers $n \geq 1$: $\begin{bmatrix}1&1\\1&0\end{bmatrix}^n=\begin{bmatrix}F_{n+1}&F_n\\F_n&F_{n-1}\end{bmatrix}$

Take determinants:

$$F_{n-1}F_{n+1}-F_n^2 = \det\begin{bmatrix}F_{n+1}&F_n\\F_n&F_{n-1}\end{bmatrix} = \det\begin{bmatrix}1&1\\1&0\end{bmatrix}^n = \left(\det\begin{bmatrix}1&1\\1&0\end{bmatrix}\right)^n = (-1)^n$$

Thus: $\boxed{F_{n-1}F_{n+1}-F_n^2=(-1)^n}$

Generalize:

$$F_{n-2}F_{n+2}-F_n^2 = \,?$$

$$F_{n-3}F_{n+3}-F_n^2 = \,?$$

$$\vdots$$

$$F_{n-k}F_{n+k}-F_n^2 = \,?$$