The Logistic Map

$f_r(x) = r \cdot x(1-x)$
$0 \le r \le 4, \quad 0 \le x \le 1$

What did you observe for $r > 3$?

At what value does a phase transition occur? 3.44? 3.45?

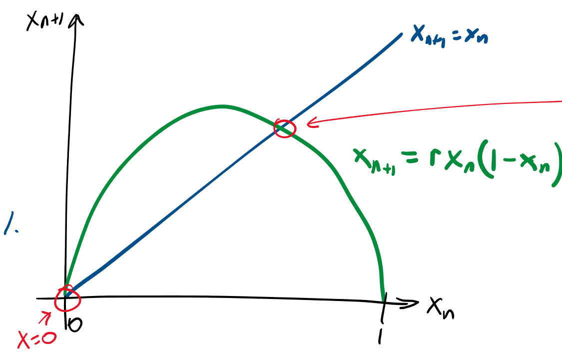
How would you describe the phase transition?

the 2-cycle splits into a 4-cycle

BIFURCATION: a qualitative change in the long-term behavior of a system

RECALL: If $r<3$, then we observe a stable fixed point.

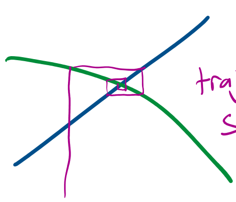
If $r<3$, then the slope of the parabola at the fixed point is between -1 and 1.



$\cancel{x} = r\cancel{x}(1-x)$
$1 = r(1-x)$
$1 = r - rx$
$rx = r - 1$
$\boxed{x = \frac{r-1}{r}}$ stable fixed point if $1 < r < 3$

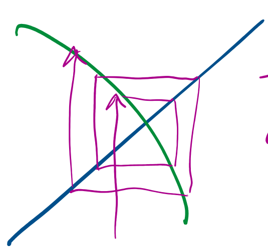
EXAMPLES: $r=2$, then fixed pt is $\frac{2-1}{2} = \frac{1}{2}$

$r = \frac{5}{2}$, then fixed pt is $\frac{\frac{5}{2}-1}{\frac{5}{2}} = \frac{\frac{3}{2}}{\frac{5}{2}} = \frac{3}{5}$



trajectory spirals in to the fixed point

If $r>3$: The slope of the parabola at the fixed point is less than -1



trajectory spirals away from the fixed point

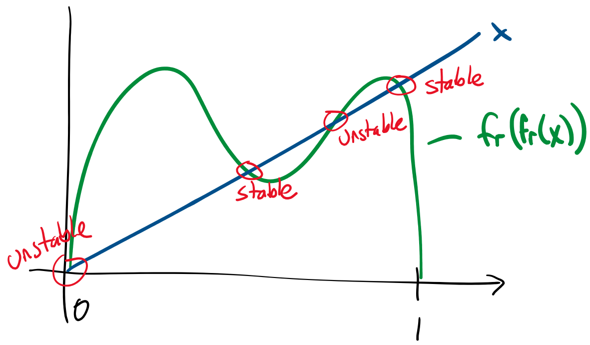
The fixed point at $x^* = \frac{r-1}{r}$ still exists, but is unstable.

If r is slightly bigger than 3, we observe a 2-cycle: oscillation between 2 values

Lets consider fixed points of $f_r(f_r(x))$.

f_r composed with itself

$$f_r(f_r(x)) = f_r\left(rx(1-x)\right) = r\left(rx(1-x)\right)\left(1-\left(rx(1-x)\right)\right)$$
$$= r^2x(1-x) - r^3x^2(1-x)^2 \quad \leftarrow \text{degree 4 polynomial}$$



The 2-cycle we observe oscillates between the 2 stable fixed points of $f_r(f_r(x))$.

At about $r = 3.45$, the 2-cycle bifurcates again and a 4-cycle appears.

Period-doubling bifurcation:

when a long-term cycle doubles its period

convergent → 2-cycle → 4-cycle → . . .