What is the shape of the prime counting function?

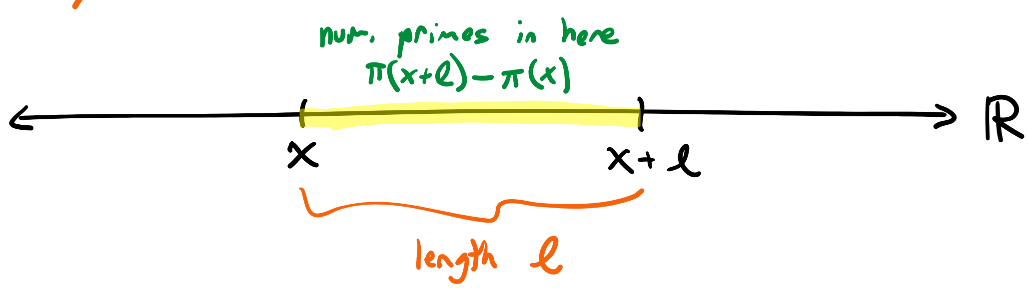
How are the prime numbers distributed?

PRIME NUMBER THEOREM

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\ln(x)}} = 1 \quad \text{and} \quad \lim_{x\to\infty} \frac{\pi(x)}{\mathrm{li}(x)} = 1$$

This limit approaches 1 faster!

Density of Primes near x:



$$\text{density at } x \approx \frac{\text{num. primes in } (x, x+\ell]}{\text{num. integers in } (x, x+\ell]}$$

$$\approx \frac{\pi(x+\ell) - \pi(x)}{\ell}$$

looks like a derivative!

differentiate

density of primes near $x \approx \dfrac{1}{\ln(x)}$ integrate

count of primes up to $x \approx \boxed{\int_0^x \frac{1}{\ln(t)}\,dt} = \mathrm{li}(x)$

LOGARITHMIC INTEGRAL

From a graph, we see that $\pi(x) < \mathrm{li}(x)$.

In 1914, John Littlewood proved that $\pi(x) - \mathrm{li}(x)$ changes sign infinitely many times.

Stanley Skews proved that the smallest x such that $\pi(x) \geq \mathrm{li}(x)$ is $x < 10^{10^{10^{34}}}$.

Computational evidence involving 10^{11} zeros of the Riemann zeta function suggests that this smallest x is approx. 1.397×10^{316}.