4 March 2024

GENERALIZED FIBONACCI SEQUENCES

Today: change the recurrence

Define: $G_0 = a$, $G_1 = b$, and $\underbrace{G_n = r \cdot G_{n-1} + s\, G_{n-2}}_{\text{recurrence}}$

for integers $n > 1$.

Here, $a, b, r,$ and s are any numbers that we like.

DEFINE: The Pell Sequence: choose $a=0$, $b=1$, $r=2$, $s=1$

$P_0 = 0$, $P_1 = 1$, and $P_n = 2P_{n-1} + P_{n-2}$ for $n > 1$.

sequence: $0, 1, 2, 5, 12, 29, 70, 169, 408, \ldots$

APPLICATION: approximating $\sqrt{2} = 1.414213\ldots$

start with $\frac{x}{y} = \sqrt{2}$ ← no integer solutions

$x = y\sqrt{2}$

$x^2 = 2y^2$

$x^2 - 2y^2 = 0$

Replace the 0 with ± 1:

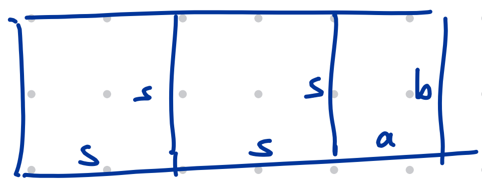
PELL'S EQUATION $\boxed{x^2 - 2y^2 = \pm 1}$ ← has integer solutions

Solutions:	$x=1$	$x=3$	$x=7$	$x=17$	$x=41$
Pell Sequence →	$y=1$	$y=2$	$y=5$	$y=12$	$y=29$
	$\frac{x}{y} = 1$	$\frac{x}{y} = 1.5$	$\frac{x}{y} = 1.4$	$\frac{x}{y} = \frac{17}{12} = 1.41\ldots$	$\frac{x}{y} = \frac{41}{29} = 1.413\ldots$

fractions converge → $\sqrt{2}$

$$17^2 - 2(12)^2$$
$$= 289 - 2(144) = 1$$

RATIOS: $\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = 1+\sqrt{2}$ silver ratio!



$$\frac{b}{a} = \frac{2s+a}{b} = 1+\sqrt{2}$$