MATH 242 — 4 March 2026

Generalized Fibonacci Sequence

"gibonacci"

$$\underbrace{G_0 = a, \quad G_1 = b,}_{\text{starting values}} \qquad \underbrace{G_n = r \cdot G_{n-1} + s \cdot G_{n-2} \text{ for } n > 1}_{\text{recurrence}}$$

We can choose any integers a, b, r, s.

Pell Sequence:

definition: $P_0 = 0, \; P_1 = 1, \quad P_n = 2P_{n-1} + P_{n-2}$

$0, 1, 2, 5, 12, 29, 70, 169, 408, \ldots$

Application: approximating $\sqrt{2} \approx 1.414213\ldots$

Since $\sqrt{2}$ is irrational, there are no integers x and y such that $\sqrt{2} = \frac{x}{y}$.

Equivalently, no integers x and y such that:

$$y\sqrt{2} = x \quad \text{or} \quad y^2 \cdot 2 = x^2$$

$$\text{or} \quad \underline{x^2 - 2y^2 = 0.}$$

What if I want integer solutions to $x^2 - 2y^2 = \pm 1$?

Solutions: $x = 1, \; y = 1$; $\quad x = 3, \; y = 2$; $\quad x = 7, \; y = 5$

$1^2 - 2(1)^2 = -1 \qquad 3^2 - 2(2)^2 = 1 \qquad 7^2 - 2(5)^2 = 49 - 50 = -1$

Approximation to $\sqrt{2}$:

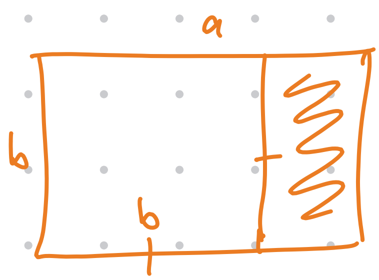
$\frac{1}{1} = 1$, $\frac{3}{2} = 1.5$, $\frac{7}{5} = 1.4$, $\frac{17}{12}$, $\frac{41}{29}$, …

sequence of fractions that converge to $\sqrt{2}$

and denominators are the Pell numbers!

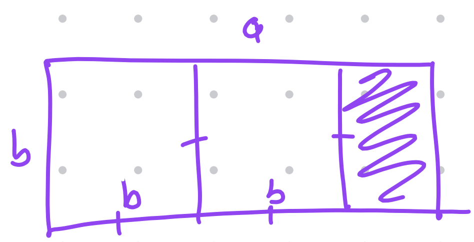
$$\lim_{n \to \infty} \frac{P_{n+m}}{P_n} = (1+\sqrt{2})^m$$

$1+\sqrt{2}$ is the <u>silver</u> ratio



$\frac{a}{b} = \frac{1+\sqrt{5}}{2}$

is the golden ratio



$\frac{a}{b} = 1+\sqrt{2}$ is the silver ratio

There is also a bronze ratio, copper ratio, etc.

Look up "metallic means" to learn more!