Scissors Congruence in 3D

MATH 261 Computational Geometry

Let P be a polyhedron and let e be an edge of P. The **dihedral angle** at edge e is the interior angle between the two faces of P that intersect along edge e.

Question 1: What is the dihedral angle along any edge of a regular tetrahedron?

Let $f : \mathbb{R} \to \mathbb{Q}$ be any function such that:

(a) $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$,

(b) $f(qx) = qf(x)$ for all $q \in \mathbb{Q}$ and $x \in \mathbb{R}$,

(c) $f(\pi) = 0$.

Question 2: If θ is any rational multiple of π, then what can you say about $f(\theta)$?

For any edge e of a polyhedron, let $\ell(e)$ be the length of e and $\phi(e)$ the dihedral angle at e. Call $\ell(e) \cdot f(\phi(e))$ the **mass** of edge e.

The **Dehn invariant** of a polyhedron P is ____.

Theorem: Let f be any d-function. If polyhedron P is dissected into $P_1, P_2, \ldots, P_n$, then

$$D_f(P) = D_f(P_1) + D_f(P_2) + \cdots + D_f(P_n).$$

Proof:

Question 3: Let P and Q be polyhedra, and let f be any d-function. If $D_f(P) \neq D_f(Q)$, then can P and Q be scissors congruent?

Question 4: How does your previous answer imply that a cube cannot be scissors congruent to a regular tetrahedron?