Math 262

1. Suppose that 45% of the phone calls you receive are scam calls. Assume that the probability of a scam call is independent from one call to the next. Let X be the number of calls that you receive until (and including) the next scam call.

   Express each of the following probabilities in terms of X. Then compute each probability.

   (a) The probability that *none* of the first 4 calls are scam calls.

   (b) The probability that none of the first 7 calls are scam calls, given that none of the first 4 calls are scam calls.

   (c) The probability that none of the first $4 + k$ calls are scam calls, given that none of the first 4 calls are scam calls.

2. Generalize your answer from problem 1(c) above. What property does this suggest for geometric random variables?

3. Let $X \sim \text{Geometric}(p)$.

   (a) Compute the mgf $M_X(t)$.

   (b) Use the infinite geometric sum formula $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ to write $M_x(t)$ without a summation.

4. Suppose random variable X has probability mass function $P(X = x) = \frac{27}{40}\left(\frac{1}{3}\right)^x$, for integers $0 \leq x \leq 3$.

   (a) Verify that this is a valid probability mass function.

   (b) Compute $E(X)$.

   (c) Find the moment generating function $M_X(t)$ of X.

   (d) Use the finite geometric sum formula $\sum_{n=0}^{m} ar^n = \frac{a(1-r^{m+1})}{1-r}$ to write the mgf without a summation.

   (e) Use technology to compute $M'_X(0)$. Does your answer agree with the expected value of X from part (b)?

5. The *skewness coefficient* of the distribution of a random variable X is

$$\gamma = \frac{E[(X-\mu)^3]}{\sigma^3}.$$

The skewness is 0 if the distribution is symmetric, positive if the distribution is skewed right, or negative if the distribution is skewed left.

(a) Expand $(X-\mu)^3$ and use this to express $E[(X-\mu)^3]$ in terms of the moments $E(X)$, $E(X^2)$, and $E(X^3)$. Then express γ in terms of these moments.

(b) For each of the following random variables, use Mathematica to compute the skewness coefficient from the mgf. Does the skewness coefficient agree with what you know about the shape of the distribution?

- $X \sim \text{Bin}(10, \frac{1}{2})$ — binomial mgf: $\left(1-p+pe^t\right)^n$

- $X \sim \text{Bin}(10, \frac{3}{4})$

- $X \sim \text{Geometric}(\frac{1}{3})$ — geometric mgf: $\dfrac{pe^t}{1-(1-p)e^t}$

- $X \sim \text{Poisson}(4)$ — Poisson mgf: $e^{\mu(e^t-1)}$

6. The monthly amount of time X (in hours) during which a manufacturing plant is inoperative due to equipment failures or power outage follows approximately a distribution with the following moment generating function:

$$M_X(t) = \left(\frac{1}{1-7.5t}\right)^2$$

The amount of loss in profit due to the plant being inoperative is given by $Y = 12X + 1.25X^2$. Determine the variance of the loss in profit. *(Actuary Exam P practice problem)*

★ **BONUS:** Let $Y = aX + b$. Differentiate both sides of $M_Y(t) = e^{tb}M(at)$ to show that $E(Y) = aE(X) + b$. Differentiate again to show that $\text{Var}(Y) = a^2\text{Var}(X)$.