$$(X_1, X_2) \underset{v_1, v_2}{\overset{u_1, u_2}{\rightleftarrows}} (Y_1, Y_2)$$

# BIVARIATE TRANSFORMATION THEOREM

Let $X_1$ and $X_2$ have joint density $f(x_1, x_2)$.

Let $Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$,

with inverse transformation $X_1 = v_1(Y_1, Y_2)$ and $X_2 = v_2(Y_1, Y_2)$.

Let $M$ be the Jacobian matrix:

$$M = \begin{bmatrix} \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial y_2} \\ \frac{\partial v_2}{\partial y_1} & \frac{\partial v_2}{\partial y_2} \end{bmatrix}$$

Then the joint density of $Y_1$ and $Y_2$ is given by

$$g(y_1, y_2) = f\Big(\underbrace{v_1(y_1, y_2)}, \underbrace{v_2(y_1, y_2)}\Big) \cdot \left|\det(M)\right|.$$

$$f(\ x_1 \qquad\qquad x_2\ )$$

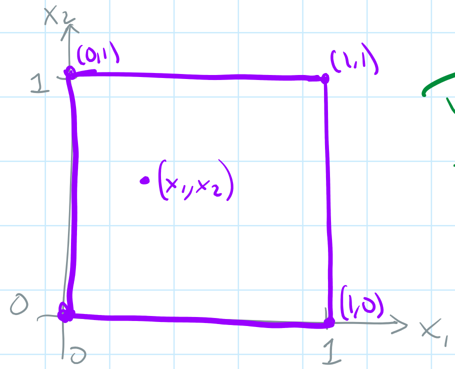
1-var theorem: $f_Y(y) = f_X(h(y)) \cdot |h'(y)|$

① $X_1, X_2 \sim \text{Unif}[0,1]$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

invert:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$




$Y_1 = X_1 + X_2$

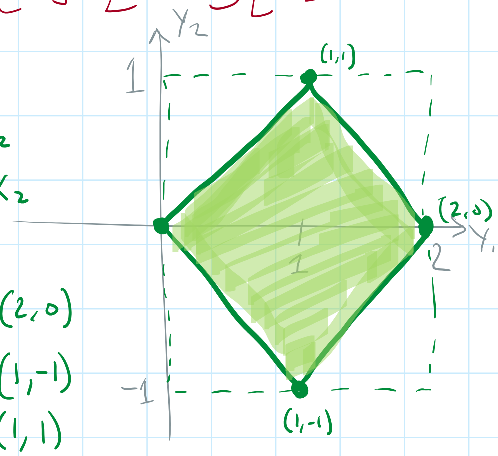
$Y_2 = X_1 - X_2$

$(1,1) \to (2,0)$

$(0,1) \to (1,-1)$

$(1,0) \to (1,1)$

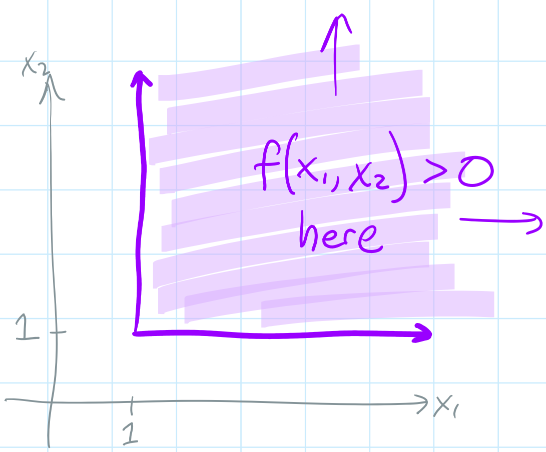
$(0,0) \to (0,0)$




$0 \le Y_1 \le 2$

$-1 \le Y_2 \le 1$

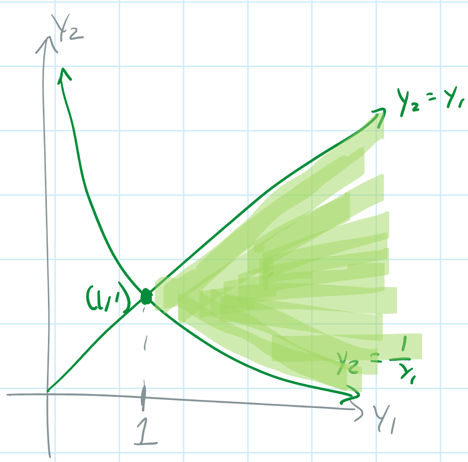
②




$Y_1 = X_1 X_2$

$Y_2 = \dfrac{X_1}{X_2}$

$Y_1 \ge 1$




Fix a value $y_1 > 1$.

$Y_2$ is smallest when $x_1 = 1$, so $Y_2 = \dfrac{1}{X_2} = \dfrac{1}{Y_1}$

$Y_1 = X_1 X_2 = 1 \cdot X_2$

$Y_2$ is largest when $x_2 = 1$, so $Y_2 = \dfrac{X_1}{1} = \dfrac{Y_1}{1}$

$Y_1 = X_1 X_2 = x_1 \cdot 1$