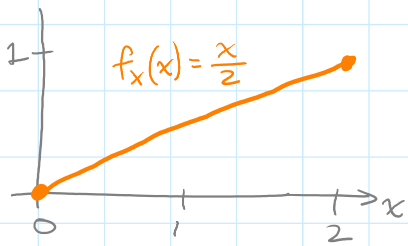
TRANSFORMATION OF A RANDOM VARIABLE

1. Let X have density $f_X(x) = \frac{x}{2}$ for $0 \le x \le 2$, and let $Y = X^2$. What is the density of Y?

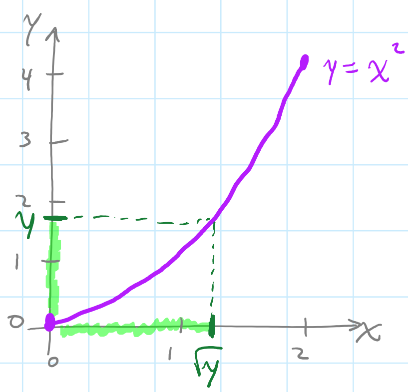


$Y = X^2$

transformation function

$y = g(x) = x^2$

Note: $0 \le Y \le 4$



Find the cdf of Y:

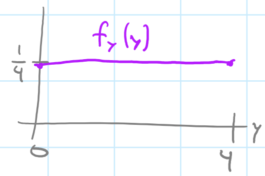
for $y \in [0,4]$:

$$F_Y(y) = P(Y \le y) = P(X^2 \le y)$$

$$= P(X \le \sqrt{y}) = \int_0^{\sqrt{y}} \frac{x}{2}\,dx = \frac{x^2}{4}\Big|_{x=0}^{x=\sqrt{y}} = \frac{(\sqrt{y})^2}{4} - \frac{0^2}{4} = \frac{y}{4}$$

Differentiate to find the pdf of Y:

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}\left(\frac{y}{4}\right) = \frac{1}{4} \quad \text{for } 0 \le y \le 4$$



Transformation Theorem:

Since $g(x) = x^2$ is strictly increasing for $0 \le x \le 2$,

the theorem says:

$$f_Y(y) = f_X(h(y))\,|h'(y)|$$

where $h(y)$ is the inverse function of $g(x)$.

For this problem: $g(x) = x^2$, so inverse is $h(y) = \sqrt{y}$

So: $$f_Y(y) = \frac{h(y)}{2} \cdot |h'(y)| = \frac{\sqrt{y}}{2} \cdot \left|\frac{1}{2\sqrt{y}}\right| = \frac{1}{4} \quad \text{for } 0 \le y \le 4$$