DENSITY OF A SUM

Let X and Y be independent random variables and $W = X + Y$.

For continuous variables: $$f_W(w) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(w-x)\, dx$$ Convolution integral

equivalently, $$= \int_{-\infty}^{\infty} f_Y(x)\, f_X(w-x)\, dx$$

For all variables: $$M_W(t) = M_X(t)\, M_Y(t)$$

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1. $W = X + Y$ $\qquad X, Y \sim \text{Unif}[0,1]$

$f_X(x) = 1$ if $0 \le x \le 1$
$f_Y(y) = 1$ if $0 \le y \le 1$

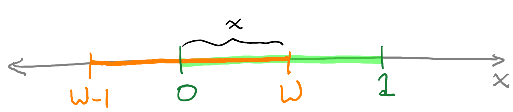
density of W: $$f_W(w) = \int_{-\infty}^{\infty} \underbrace{f_X(x)}_{\text{is } 1} \; \underbrace{f_Y(w-x)}_{\text{is } 1 \text{ if } 0 \le w-x \le 1}\, dx$$

is 1 if $0 \le x \le 1$

is 1 if $0 \le w - x \le 1$
$-w \le -x \le 1 - w$
$w - 1 \le x \le w$

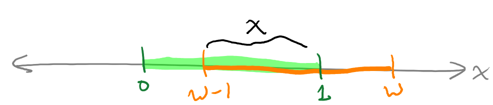
If both conditions are true, then the integrand is 1; otherwise 0.

If $0 \le w \le 1$:



$$f_W(w) = \int_0^w 1 \cdot 1\, dx = w$$

If $1 \le w \le 2$:



$$f_W(w) = \int_{w-1}^{1} 1 \cdot 1\, dx = x \Big|_{w-1}^{1} = 1 - (w-1) = 2 - w$$

So: $$f_W(w) = \begin{cases} w & \text{if } 0 \le w \le 1, \\ 2 - w & \text{if } 1 < w \le 2. \end{cases}$$

2. $Z = X_1 + X_2 + X_3 = X_1 + W$

$$f_z(z) = \int_{-\infty}^{\infty} f_x(x)\, f_W(z-x)\, dx$$