BIVARIATE TRANSFORMATION THEOREM

Let X_1 and X_2 have joint density $f(x_1, x_2)$.

Let $Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$,

with inverse transformation $X_1 = v_1(Y_1, Y_2)$ and $X_2 = v_2(Y_1, Y_2)$.

Let M be the Jacobian matrix:

$$\begin{tikzcd} (X_1, X_2) \arrow[r, bend left, "u_1, u_2"] & (Y_1, Y_2) \arrow[l, bend left, "v_1, v_2"] \end{tikzcd}$$

$$M = \begin{bmatrix} \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial y_2} \\ \frac{\partial v_2}{\partial y_1} & \frac{\partial v_2}{\partial y_2} \end{bmatrix}$$

Then the joint density of Y_1 and Y_2 is given by

$$g(y_1, y_2) = f(v_1(y_1, y_2), v_2(y_1, y_2)) \cdot |\det(M)|.$$

$$f(\quad x_1 \quad , \quad x_2 \quad)$$

Note similarity to 1-variable Transformation Theorem!

$$f_Y(y) = f_X(h(y)) \cdot |h'(y)|$$