1. Suppose that during a meteor shower, ten visible meteors per hour are expected.

(a) Let X be the number of visible meteors in one hour. What assumptions must we make in order to say that X has a Poisson distribution?

We must assume that meteors occur independently, and that the average rate over time is known (eg. ten per hour).

(b) What is the probability that $5 \le X \le 15$?

If $X \sim \text{Poisson}(10)$, then $P(5 \le X \le 15) \approx 0.922$

R: ppois(15, 10) − ppois(4, 10)

Mathematica: CDF[PoissonDistribution[10], 15] − CDF[PoissonDistribution[10], 4] // N

2. Suppose that the number of phone calls an office receives has a Poisson distribution with a mean of 5 calls per hour.

(a) What is the probability that exactly 7 calls are received between 10:00 and 11:00?

$X \sim \text{Poisson}(5)$ $\qquad P(X = 7) = e^{-5}\frac{5^7}{7!} \approx 0.104$

R: dpois(7, 5)

Mathematica: PDF[PoissonDistribution[5], 7]

(b) What is the probability that more than 7 calls are received between 10:00 and 11:00?

$P(X > 7) = 1 - P(X \le 7) \approx 0.133$

R: 1 − ppois(7, 5)

Mathematica: 1 − CDF[PoissonDistribution[5], 7] // N

(c) What is the probability that exactly 10 calls are received between 10:00 and 12:00?

In two hours, the mean number of calls received is ten.

Let $Y \sim \text{Poisson}(10)$. Then $P(Y = 10) = e^{-10}\frac{10^{10}}{10!} \approx 0.125$

R: dpois(10, 10)

Mathematica: PDF[PoissonDistribution[10], 10] // N

3. Suppose that a machine produces items, 2% of which are defective. Let X be the number of defective items among 500 randomly-selected items produced by the machine.

(a) What is the distribution of X?

$$X \sim \text{Bin}(500, 0.02)$$

(b) What are the mean and variance of X?

$$E(X) = \underset{n}{500}(\underset{p}{0.02}) = 10, \qquad Var(X) = \underset{n}{500}(\underset{p}{0.02})(\underset{1-p}{0.98}) = 9.8$$

(c) What is $P(X = 12)$?

$$P(X=12) = \binom{500}{12}(0.02)^{12}(0.98)^{488} = 0.0955$$

[If n is big (say $n \geq 100$) and p is small (say $np \leq 10$) then $\text{Bin}(n,p)$ is well-approximated by $\text{Poisson}(np)$.]

(d) What Poisson distribution approximates the distribution of X?

$\text{Bin}(500, 0.02)$ can be approximated by $\text{Poisson}(10)$.

(e) Use your Poisson distribution to approximate $P(X = 12)$?

Let $Y \sim \text{Poisson}(10)$.

Then $P(X=12) \approx P(Y=12) = e^{-10}\dfrac{10^{12}}{12!} \approx 0.0948 \leftarrow$ This is close to the answer in part (c).

4. Let $X \sim \text{Poission}(\mu)$. Show that $P(X = k)$ increaess monotonically and then decreases monotonically as k increases, reaching its maximum when k is the largest integer less than or equal to μ.

Consider the ratio of probabilities of consecutive values k and $k-1$:

$$\frac{P(X=k)}{P(X=k-1)} = \frac{e^{-\mu}\frac{\mu^k}{k!}}{e^{-\mu}\frac{\mu^{k-1}}{(k-1)!}} = \frac{\mu^k \, k!}{\mu^{k-1}\,(k-1)!} = \frac{\mu}{k}$$

If $k < \mu$, then $\frac{\mu}{k} > 1$, so $P(X=k) > P(X=k-1)$, and the sequence of probabilities increases.

If $k = \mu$ (only possible if μ is an integer), then $\frac{\mu}{k} = 1$, so $P(X=k) = P(X=k-1)$.

If $k > \mu$, then $\frac{\mu}{k} < 1$, so $P(X=k) < P(X=k-1)$, and the sequence of probabilities decreases.

Thus, the max value of $P(X=k)$ occurs when k is the largest integer less than or equal to μ. The sequence of probabilities increases up to this value and decreases afterward.