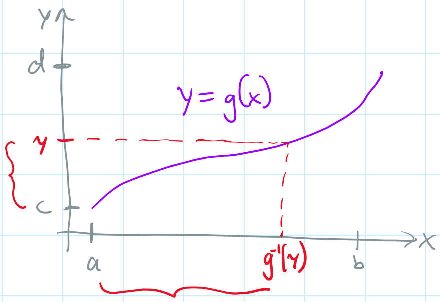
FROM LAST TIME:

If X has pdf $f_X(x)$ and $Y = g(X)$

(suppose $a \le X \le b$ and g is an increasing function)



Then: $F_Y(y) = P(Y \le y) = P(g(X) \le y)$

$$= P(X \le g^{-1}(y))$$

$$F_Y(y) = P(X \le h(y)) = F_X(h(y)) \quad \text{where } h = g^{-1}$$

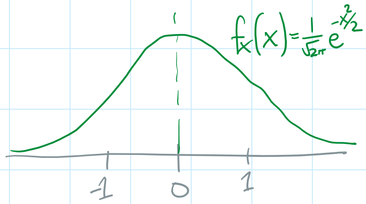
Differentiate:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(h(y))$$ chain rule

$$f_Y(y) = f_X(h(y)) \cdot h'(y)$$

TRANSFORMATION THEOREM

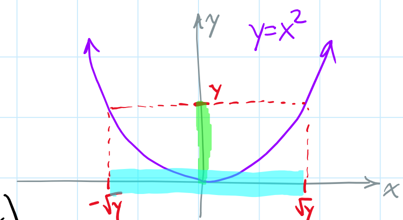
#4 FROM LAST TIME: $X \sim N(0,1)$ standard normal distribution



let: $Y = X^2$

then $Y \ge 0$

Find $F_Y(y)$:



Let $y \ge 0$: $F_Y(y) = P(Y \le y)$

$$= P(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx$$

$$F_Y(y) = 2\int_0^{\sqrt{y}} f_X(x)\,dx \quad \text{by symmetry of } f_X \text{ about zero}$$

From Calculus: FTC 2

$$\frac{d}{dy}\int_c^{g(y)} f(x)\,dx = f(g(y)) \cdot g'(y)$$

Differentiate: $\frac{d}{dy} F_Y(y) = \frac{d}{dy} 2\int_0^{\sqrt{y}} f_X(x)\,dx$

$$= 2 f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}}$$

$$= \cancel{2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{y})^2}{2}} \cdot \frac{1}{\cancel{2}\sqrt{y}}$$

$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/2} \quad \text{for } y > 0}$$

SIMULATION OF CONTINUOUS RANDOM VARIABLES

Inverse cdf method:

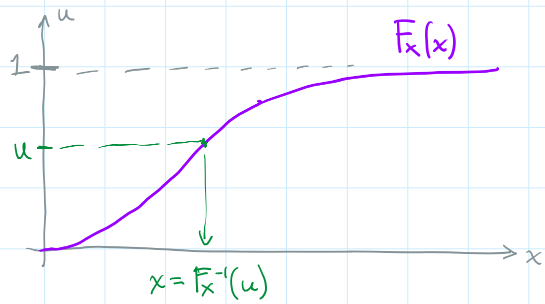
To simulate values of X with pdf $f_X(x)$:

Find cdf F_X and its inverse F_X^{-1}.

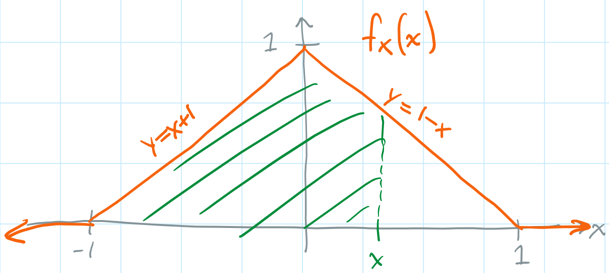
Generate a value $U \sim \text{Unif}[0,1]$.

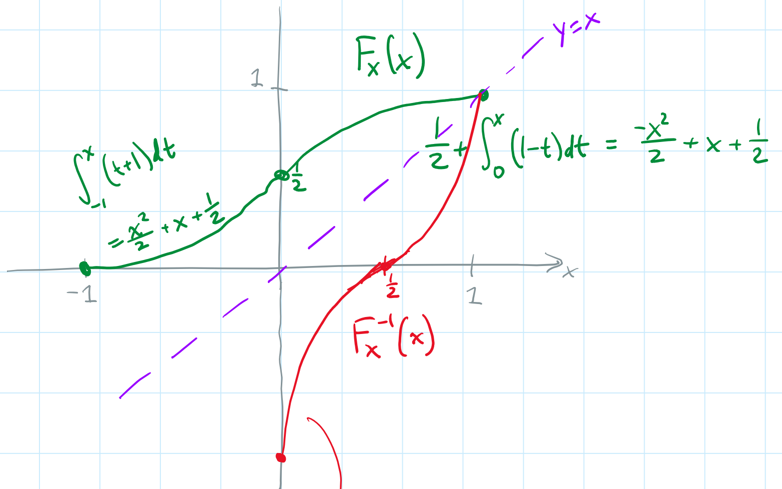
Then take $X = F_X^{-1}(U)$.

transformation



1 $f_X(x) = \begin{cases} x+1 & \text{if } -1 \le x \le 0 \\ 1-x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$





Set $u = \frac{x^2}{2} + x + \frac{1}{2}$

and solve for x:

$$0 = \frac{x^2}{2} + x + \left(\frac{1}{2} - u\right)$$

$$0 = x^2 + 2x + (1 - 2u)$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1-2u)}}{2} = \frac{-2 \pm \sqrt{4 - 4 + 8u}}{2} = \frac{-2 \pm 2\sqrt{2u}}{2}$$

$$x = -1 \pm \sqrt{2u} \qquad \text{if} \quad 0 \le u \le \frac{1}{2}$$

Choose the + case since we need $x > 0$

Similarly for $\frac{1}{2} < u < 1$.

The inverse of $u = F_X(x)$ is: $F_X^{-1}(u) = \begin{cases} -1 + \sqrt{2u} & \text{if } 0 \le u \le \frac{1}{2} \\ 1 - \sqrt{2-2u} & \text{if } \frac{1}{2} < u \le 1 \end{cases}$