1. A cafeteria has three meal options: pizza, burgers, and salad bar. Three students each choose one option independently at random (equally likely to choose any option). Let X be the number (of the 3) who choose pizza, and let Y be the number who choose the salad bar.

(a) Compute the following probabilities

$P(X = 0 \text{ and } Y = 0)$ $= P\left(\begin{matrix}\text{student 1}\\ \text{chooses burger}\end{matrix}\right) \cdot P\left(\begin{matrix}\text{student 2}\\ \text{chooses burger}\end{matrix}\right) \cdot P\left(\begin{matrix}\text{student 3}\\ \text{chooses burger}\end{matrix}\right) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27}$

3 possible students who could choose salad

$P(X = 0 \text{ and } Y = 1)$ There are 3 ways that one student can choose salad while the other two choose burger. Thus:

$P(X=0 \text{ and } Y=1) = 3 \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{3}{27} = \frac{1}{9}$

(b,c) What is the joint pmf of X and Y? What are the marginal pmfs of X and Y?

1 2 3
PSB
PBS
BPS
BSP
SBP
SPB

six arrangements of the 3 choices

joint pmf:

Y \ X	0	1	2	3	marginal pmf: $p_Y(y)$
0	$\frac{1}{27}$	$\frac{3}{27}$	$\frac{3}{27}$	$\frac{1}{27}$	$\frac{8}{27}$
1	$\frac{3}{27}$	$\frac{6}{27}$	$\frac{3}{27}$	0	$\frac{12}{27}$
2	$\frac{3}{27}$	$\frac{3}{27}$	0	0	$\frac{6}{27}$
3	$\frac{1}{27}$	0	0	0	$\frac{1}{27}$
marginal pmf: $p_X(x)$	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$	

(d) Are X and Y independent? Why or why not?

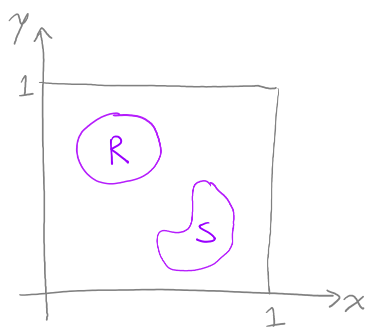
No, since knowledge of one affects the probabilities of the other.

For example: If you know that $X=3$, then it must be that $Y=0$.

2. Suppose a particle is randomly located in the square $0 \le x \le 1$, $0 \le y \le 1$. Let (X, Y) be the coordinates of the particle.

(a) Is $f(x, y) = 1$ for $0 \le x \le 1$ and $0 \le y \le 1$ a reasonable joint density function for X and Y? Why or why not?

Yes, this is reasonable. If R and S are regions of the same area within the unit square, then the particle is equally likely to be in R or S.



This means that the joint density $f(x,y)$ must satisfy

$$\iint_R f(x,y)\,dA = \iint_S f(x,y)\,dA$$

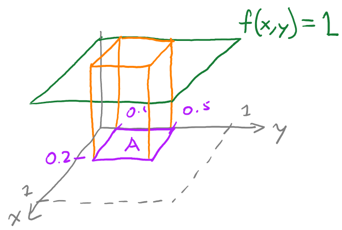
whenever R and S have equal areas.

The only way to achieve this is for $f(x,y)$ to be constant on the unit square.

So $f(x,y) = k$ for $0 \le x \le 1$ and $0 \le y \le 1$, and $f(x,y) = 0$ otherwise.

Since $\int_0^1\int_0^1 k\,dx\,dy = k = 1$, we have $f(x,y) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$

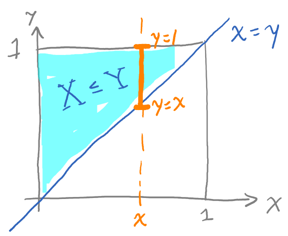
(b) Find $P(X \le 0.2, 0.1 \le Y \le 0.5)$.



$$P(X \le 0.2,\ 0.1 \le Y \le 0.5) = \int_{0.1}^{0.5}\int_0^{0.2} f(x,y)\,dx\,dy$$

$$= (0.2)(0.4)(1) = 0.08$$

(c) Find $P(X \le Y)$.



$$P(X \le Y) = \iint_A 1\,dA = \frac{1}{2}$$

$$= \int_0^1\int_x^1 1\,dy\,dx$$

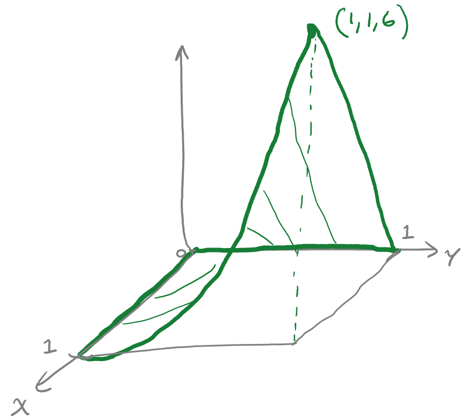
(d) Are X and Y independent? Why or why not?

Yes: $f(x,y) = f_X(x)\,f_Y(y)$ for $0 \le x \le 1$, $0 \le y \le 1$.

We didn't get to this in class.

3. Let X and Y have joint pdf $f(x,y) = 6xy^2$ for $0 \le x \le 1$ and $0 \le y \le 1$.

(a) Verify that $f(x,y)$ is a joint pdf.



$f(x,y) \ge 0$ for $0 \le x \le 1$, $0 \le y \le 1$ and

$$\int_0^1\int_0^1 6xy^2\,dx\,dy = 6\int_0^1 x\,dx \int_0^1 y^2\,dy$$
$$= 6\left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right) = 1$$

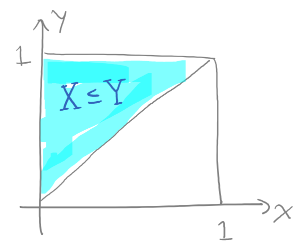
(b) What is $f_X(x)$?

$$f_X(x) = \int_0^1 6xy^2\,dy = 2xy^3\Big|_{y=0}^{y=1} = 2x \quad \text{for } 0 \le x \le 1$$

(c) What is $P(X \le Y)$?

$$P(X \le Y) = \int_0^1\int_x^1 6xy^2\,dy\,dx = \int_0^1 (2x - 2x^4)\,dx = 1 - \tfrac{2}{5} = \tfrac{3}{5}$$

$$\int_x^1 6xy^2\,dy = 2xy^3\Big|_{y=x}^{y=1} = 2x - 2x^4$$



(d) Are X and Y independent? Why or why not?

$$f_Y(y) = \int_0^1 6xy^2\,dx = 3x^2y^2\Big|_{x=0}^{x=1} = 3y^2$$

Yes: $f(x,y) = f_X(x)\,f_Y(y)$

$6xy^2 = (2x)(3y^2)$ for $0 \le x \le 1$, $0 \le y \le 1$