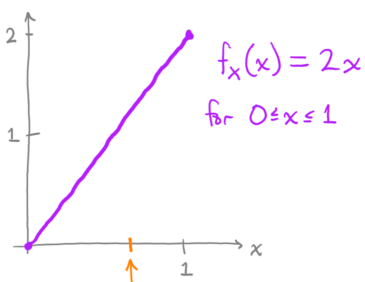
1. Let X and Y have joint pdf $f(x,y) = 6xy^2$ for $0 \le x \le 1$ and $0 \le y \le 1$.

(a) Find the marginal pdfs $f_X(x)$ and $f_Y(y)$.

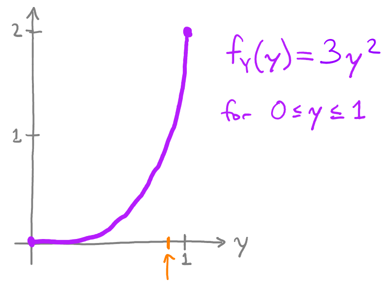
$$f_X(x) = \int_0^1 6xy^2\,dy = 2xy^3\Big|_{y=0}^{y=1} = 2x \quad \text{for } 0 \le x \le 1$$

$$f_Y(y) = \int_0^1 6xy^2\,dx = 3x^2y^2\Big|_{x=0}^{x=1} = 3y^2 \quad \text{for } 0 \le y \le 1$$

(b) Sketch the marginal pdfs $f_X(x)$ and $f_Y(y)$. What would you estimate to be the means $E(X)$ and $E(Y)$?



$E(X)$ is about here
maybe $E(X) \approx 0.7$?



$E(Y)$ should be close to 1
maybe $E(Y) \approx \frac{7}{8}$?

(c) Compute $E(X)$ and $E(Y)$.

$f_X(x) = 2x$, $0 \le x \le 1$, so $\quad E(X) = \int_0^1 x \cdot 2x\,dx = \frac{2}{3}x^3\Big|_0^1 = \boxed{\frac{2}{3}}$

$f_Y(y) = 3y^2$, $0 \le y \le 1$, so $\quad E(Y) = \int_0^1 y \cdot 3y^2\,dy = \frac{3}{4}y^4\Big|_0^1 = \boxed{\frac{3}{4}}$

(d) Compute $E(X+Y)$ in two different ways.

I. Using linearity of expected value:

$$E(X+Y) = E(X) + E(Y) = \frac{2}{3} + \frac{3}{4} = \frac{17}{12}$$

II. Using the formula for bivariate expected value:

$$E(X+Y) = \int_0^1\int_0^1 (x+y)\,6xy^2\,dy\,dx = \boxed{\frac{17}{12}}$$

Mathematica: `Integrate[(x + y) 6 x y^2, {x, 0, 1}, {y, 0, 1}]`

$\underbrace{\{x, 0, 1\}}_{\text{outer bounds}}$, $\underbrace{\{y, 0, 1\}}_{\text{inner bounds}}$

(e) How many ways are there to compute $E(XY)$? Now compute $E(XY)$.

There is only one way to do this:

$$E(XY) = \int_0^1 \int_0^1 (xy)\, 6xy^2 \, dy\, dx = \boxed{\frac{1}{2}}$$

Note that for this problem, $E(XY) = E(X)E(Y)$.

(f) What are the values of $\text{Cov}(X, Y)$ and $\text{Corr}(X, Y)$?

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{2} - \frac{2}{3}\cdot\frac{3}{4} = 0$$

$$\text{Corr}(X,Y) = 0$$

2. Let X and Y have joint pdf $f(x, y) = 3x + 3y$ for $0 \le x$, $0 \le y$, and $x + y \le 1$.

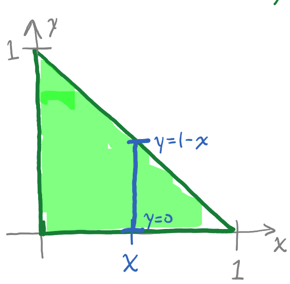
(a) Sketch the joint pdf and verify that the volume underneath is 1.

volume:

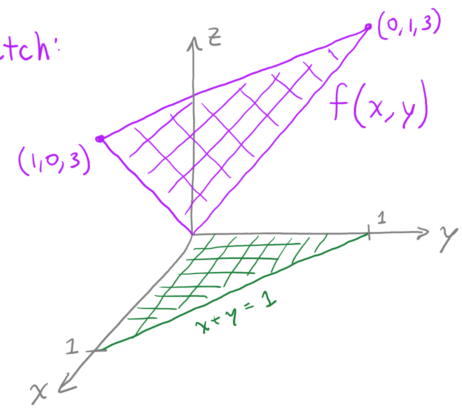
$$\int_0^1 \int_0^{1-x} (3x+3y)\, dy\, dx = 1$$

note the bounds of integration

$0 \le x \le 1, \quad 0 \le y \le 1-x$



sketch:



To compute the integral using Mathematica:

```
Integrate[3 x + 3 y, {x, 0, 1}, {y, 0, 1 - x}]
```

outer bounds — inner bounds

(b) What values of X and Y have high probability? What values have low probability?

X and Y are likely to have a sum close to 1.

X and Y are not likely to both be near zero.

(c) Compute the following, using technology to evaluate integrals:

- $E(X+Y)$ $\quad E(X+Y) = \int_0^1 \int_0^{1-x} (x+y)(3x+3y)\, dy\, dx = \boxed{\frac{3}{4}}$

```
Integrate[(x + y) (3 x + 3 y), {x, 0, 1}, {y, 0, 1 - x}]
```

- $E(XY)$ $\quad E(XY) = \int_0^1 \int_0^{1-x} (xy)(3x+3y)\, dy\, dx = \boxed{\frac{1}{10}}$

```
Integrate[x y (3 x + 3 y), {x, 0, 1}, {y, 0, 1 - x}]
```

- $E(X)$ $\quad f_X(x) = \int_0^{1-x} (3x+3y)\, dy = \frac{3}{2}(1-x^2), \quad 0 \le x \le 1$

```
Integrate[3 x + 3 y, {y, 0, 1 - x}] // Simplify
```

$$E(X) = \int_0^1 x \cdot \frac{3}{2}(1-x^2)\, dx = \boxed{\frac{3}{8}}$$

```
Integrate[x (3/2) (1 - x^2), {x, 0, 1}]
```

- $E(Y)$ $\quad f_Y(y) = \int_0^{1-y} (3x+3y)\, dx = \frac{3}{2}(1-y^2), \quad 0 \le y \le 1,$

```
Integrate[3 x + 3 y, {x, 0, 1 - y}] // Simplify
```

$$E(Y) = \int_0^1 y \cdot \frac{3}{2}(1-y^2)\, dy = \boxed{\frac{3}{8}}$$

```
Integrate[y (3/2) (1 - y^2), {y, 0, 1}]
```

(d) What is $\mathrm{Cov}(X, Y)$?

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{10} - \frac{3}{8} \cdot \frac{3}{8} = \boxed{\frac{-13}{320}}$$

3. How do $E(X)$ and $E(Y)$ relate to $E(X+Y)$ and $E(XY)$? Does independence play a role?

$E(X+Y) = E(X) + E(Y)$ by linearity of expectation.

If X and Y are independent, then $E(XY) = E(X)E(Y)$.

(The converse is not true!)