

1. Let $X \sim \text{Unif}[-1,1]$ and $Y = X^2$.

(a) Compute $E(X)$, $E(Y)$, and $E(XY)$.

$$E(X) = 0$$

$$E(Y) = E(X^2) = \int_{-1}^1 x^2 \cdot \frac{1}{2} dx = \frac{1}{6} x^3 \Big|_{-1}^1 = \frac{1}{3}$$

$$E(XY) = E(X^3) = \int_{-1}^1 x^3 \cdot \frac{1}{2} dx = \frac{1}{8} x^4 \Big|_{-1}^1 = 0$$

(b) Are X and Y independent? Are X and Y correlated? Why or why not?

X and Y are dependent since $Y = X^2$

X and Y are uncorrelated since $E(XY) = E(X)E(Y)$

- I. Two standard, fair dice are rolled. Let X_1 and X_2 be the numbers that appear on the dice.
- II. An urn contains balls labeled 1, 2, 3, 4, 5, 6. Let Y_1 and Y_2 be the numbers on two balls drawn without replacement from the urn.

2. What is the distribution of X_1 and X_2 ? How about the distribution of Y_1 and Y_2 ?

$X_1, X_2, Y_1,$ and Y_2 are all uniformly distributed on $\{1, 2, 3, 4, 5, 6\}$.

3. What are the expected values of X_1, X_2, Y_1, Y_2 ? What are the variances of these random variables?

$$E(X_1) = \frac{7}{2} \quad E(X_1^2) = \frac{1}{6} (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) = \frac{91}{6}$$

$$\text{Var}(X_1) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

Since $X_1, X_2, Y_1,$ and Y_2 all have the same distribution, they have the same expected values and variances.

4. What are $E(X_1 + X_2)$ and $\text{Var}(X_1 + X_2)$?

By linearity, $E(X_1 + X_2) = E(X_1) + E(X_2) = 7$

Since X_1 and X_2 are independent,

$$\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2) = \frac{35}{6}$$

5. What are $E(Y_1 + Y_2)$ and $\text{Var}(Y_1 + Y_2)$?

By linearity, $E(Y_1 + Y_2) = E(Y_1) + E(Y_2) = 7$

Since Y_1 and Y_2 are dependent:

$$\begin{aligned} \text{Var}(Y_1 + Y_2) &= \text{Var}(Y_1) + \text{Var}(Y_2) + 2 \text{Cov}(Y_1, Y_2) \\ &= \frac{35}{12} + \frac{35}{12} + 2\left(\frac{-7}{12}\right) = \frac{56}{12} = \frac{14}{3} \end{aligned}$$

$$\text{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2) = \frac{35}{3} - \frac{7}{2} \cdot \frac{7}{2} = -\frac{7}{12}$$

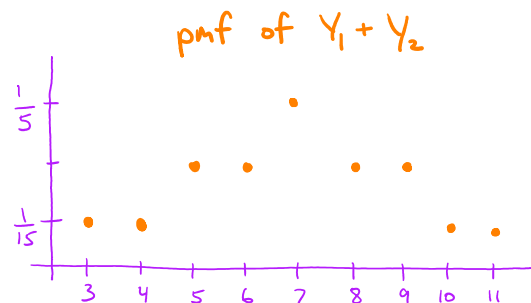
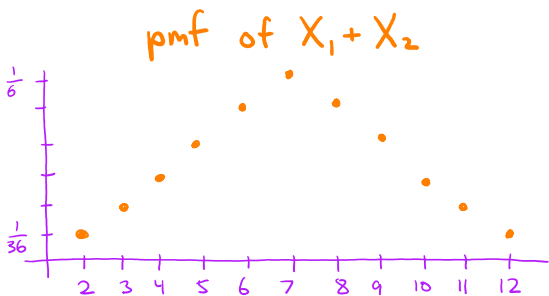
Possible products:

	1	2	3	4	5	6
1	□	2	3	4	5	6
2	2	□	6	8	10	12
3	3	6	□	12	15	18
4	4	8	12	□	20	24
5	5	10	15	20	□	30
6	6	12	18	24	30	□

← All equally likely.

$$\begin{aligned} E(Y_1 Y_2) &= \frac{1}{15} (2+3+4+5+6+6+8+10+12+12+15+18+20+24+30) \\ &= \frac{175}{15} = \frac{35}{3} \end{aligned}$$

6. Sketch the pmfs of $X_1 + X_2$ and $Y_1 + Y_2$. How does this help make sense of the means and variances that you found for these sums?



7. Generalize to rolls of n dice: find $E(X_1 + \dots + X_n)$ and $\text{Var}(X_1 + \dots + X_n)$.

$$E(X_1 + \dots + X_n) = E(X_1) + \dots + E(X_n) = \frac{7n}{2}$$

$$\text{by independence, } \text{Var}(X_1 + \dots + X_n) = \text{Var}(X_1) + \dots + \text{Var}(X_n) = \frac{35n}{12}$$

8. Similarly, generalize to choosing n balls from the urn. Find $E(Y_1 + \dots + Y_n)$ and $\text{Var}(Y_1 + \dots + Y_n)$.

Now $n \leq 6$.

$$\text{As before, } E(Y_1 + \dots + Y_n) = \frac{7n}{2}$$

$$\begin{aligned} \text{However, now } \text{Var}(Y_1 + \dots + Y_n) &= \sum_{i=1}^n \text{Var}(Y_i) + 2 \sum_{i < j} \text{Cov}(Y_i, Y_j) \\ &= \frac{35n}{12} + 2 \frac{n^2 - n}{2} \left(-\frac{7}{12}\right) = \frac{35n - 7n^2 + 7n}{12} = \frac{42n - 7n^2}{12} = \frac{7n(6-n)}{12} \end{aligned}$$

Note that if $n=6$, the variance is zero.