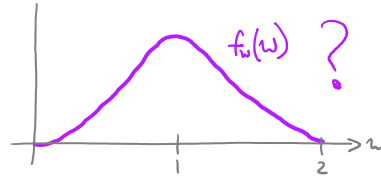


1. Let X and Y be independent uniform variables on $[0, 1]$, and let $W = X + Y$.

(a) What do you think the pdf of W will look like? Make a guess. Draw a sketch.

We know $0 \leq W \leq 2$.

Possibly the pdf of W looks like:

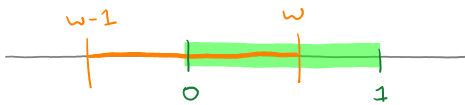


(b) Use convolution to find a formula for the pdf of W .

$$f_W(w) = \int_{-\infty}^{\infty} \underbrace{f_X(x)}_{\substack{\uparrow \\ f_X(x) = 1 \\ \text{if } 0 \leq x \leq 1}} \underbrace{f_Y(w-x)}_{\substack{\uparrow \\ f_Y(w-x) = 1 \\ \text{if } 0 \leq w-x \leq 1 \\ \text{if } w-1 \leq x \leq w}} dx$$

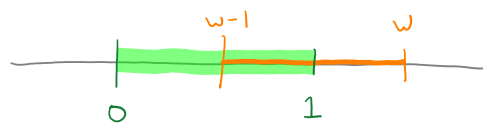
If both conditions are true, then integrand is 1; else integrand is 0.

If $0 \leq w \leq 1$:



$$f_W(w) = \int_0^w 1 dx = w$$

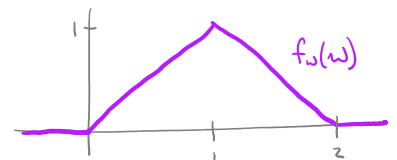
If $1 \leq w \leq 2$:



$$f_W(w) = \int_{w-1}^1 1 dx = w - (w-1) = 2 - w$$

The density of W is:

$$f_W(w) = \begin{cases} w & \text{if } 0 \leq w \leq 1 \\ 2-w & \text{if } 1 \leq w \leq 2 \\ 0 & \text{otherwise} \end{cases}$$



2. Use convolution to write an integral that gives the pdf of the sum of three independent $\text{Unif}[0,1]$ random variables. How could you evaluate the integral?

Let $T = X_1 + X_2 + X_3$, $X_i \sim \text{Unif}[0,1]$

$$f_T(t) = \int_0^3 f_X(x) f_W(t-x) dx$$

\uparrow
 $\text{Unif}[0,1]$ pdf

\uparrow pdf of $X+Y$ from #1(b)

Piecewise functions make this tricky to integrate. Use Mathematica.

If you want to do the integral by hand, here are the details:

$$f_T(t) = \begin{cases} \int_0^t (t-x) dx = \frac{1}{2}t^2, & \text{if } 0 \leq t < 1 \\ \int_0^{t-1} (2-t+x) dx + \int_{t-1}^1 (t-x) dx = 3t - t^2 - \frac{3}{2}, & \text{if } 1 \leq t < 2 \\ \int_{t-2}^1 (2-t+x) dx = \frac{1}{2}t^2 - 3t + \frac{9}{2}, & \text{if } 2 \leq t \leq 3 \end{cases}$$

3. Let $X_k \sim N(k, 1)$ for $k \in \{1, 2, \dots, m\}$, and suppose all of the X_k are independent.

(a) What is the distribution of $X_1 + X_2 + \dots + X_m$?

$N(\mu, \sigma)$ has mgf $\exp(\mu t + \sigma^2 t^2 / 2)$

$$M_{X_k}(t) = \exp\left(kt + \frac{t^2}{2}\right)$$

$$M_{X_1 + \dots + X_m}(t) = \exp\left(t + \frac{t^2}{2}\right) \exp\left(2t + \frac{t^2}{2}\right) \dots \exp\left(mt + \frac{t^2}{2}\right) = \exp\left((1+2+\dots+m)t + m \frac{t^2}{2}\right) = \exp\left(\frac{m(m+1)}{2}t + m \frac{t^2}{2}\right)$$

$$\text{Thus, } X_1 + \dots + X_m \sim N\left(\frac{m(m+1)}{2}, \sqrt{m}\right).$$

(b) What is the distribution of $X_1 + 2X_2 + \dots + mX_m$?

$$M_{kX_k}(t) = M_{X_k}(kt) = \exp\left(k^2 t + \frac{k^2 t^2}{2}\right)$$

$$\begin{aligned} M_{X_1 + 2X_2 + \dots + mX_m}(t) &= \exp\left(t + \frac{t^2}{2}\right) \exp\left(4t + \frac{4t^2}{2}\right) \dots \exp\left(m^2 t + \frac{m^2 t^2}{2}\right) = \exp\left((1+4+\dots+m^2)t + (1+4+\dots+m^2) \frac{t^2}{2}\right) \\ &= \exp\left(S t + S \frac{t^2}{2}\right), \quad \text{where } S = 1+4+\dots+m^2 = \frac{m(m+1)(2m+1)}{6} \end{aligned}$$

so the sum is $N(S, \sqrt{S})$.

4. Use moment generating functions to justify the following statements.

(a) The sum of n independent exponential random variables with common parameter λ has a gamma distribution with parameters $\alpha = n$ and $\beta = \frac{1}{\lambda}$.

Let X_1, X_2, \dots, X_n be independent $\text{Exp}(\lambda)$ random variables and $Y = X_1 + X_2 + \dots + X_n$.

exponential mgf: $M_{X_i}(t) = \frac{\lambda}{\lambda - t}$ for $t < \lambda$

Then $M_Y(t) = M_{X_1}(t) M_{X_2}(t) \dots M_{X_n}(t)$
 $= \left(\frac{\lambda}{\lambda - t}\right) \left(\frac{\lambda}{\lambda - t}\right) \dots \left(\frac{\lambda}{\lambda - t}\right) = \left(\frac{\lambda}{\lambda - t}\right)^n = \frac{1}{\left(1 - \frac{t}{\lambda}\right)^n}$

mgf of
Gamma($\alpha = n, \beta = \frac{1}{\lambda}$)

Thus, $Y \sim \text{Gamma}(\alpha = n, \beta = \frac{1}{\lambda})$.

(b) The sum of n independent geometric random variables with common parameter p has a negative binomial distribution with parameters $r = n$ and p .

Let X_1, X_2, \dots, X_n be independent $\text{Geom}(p)$ random variables
and $Y \sim X_1 + X_2 + \dots + X_n$.

Geometric mgf: $M_{X_i}(t) = \frac{pe^t}{1 - (1-p)e^t}$

Then $M_Y(t) = M_{X_1}(t) M_{X_2}(t) \dots M_{X_n}(t) = \left(\frac{pe^t}{1 - (1-p)e^t}\right)^n$,

which is the mgf of a negative binomial distribution with parameters $r = n$ and p .