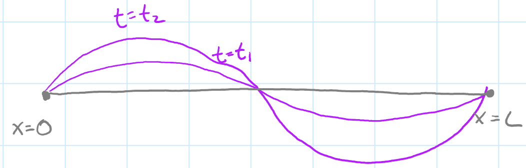
WAVE EQUATION

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$



Fixed boundaries: $u(0,t) = u(L,t) = 0$

Separation of Variables: $u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left[a_n \cos\left(\frac{cn\pi}{L}t\right) + b_n \sin\left(\frac{cn\pi}{L}t\right)\right]$

D'Alembert's Solution: $u(x,t) = p(\underbrace{x+ct}) + q(\underbrace{x-ct})$

"characteristics"

p, q are any functions

Initial position: $u(x,0) = f(x)$

Initial velocity: $\frac{\partial u}{\partial t}(x,0) = g(x)$

Last time:

$$p(x) = \frac{1}{2}f(x) + \frac{1}{2c}G(x)$$

$$q(x) = \frac{1}{2}f(x) - \frac{1}{2c}G(x)$$

where $G'(x) = g(x)$

(G is an antiderivative of g)

Thus:

$$u(x,t) = \frac{1}{2}f(x+ct) + \frac{1}{2c}G(x+ct) + \frac{1}{2}f(x-ct) - \frac{1}{2c}G(x-ct)$$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\left[G(x+ct) - G(x-ct)\right]$$

FTC: $\int_a^b g(t)\,dt = G(b) - G(a)$ where $G' = g$

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau}$$

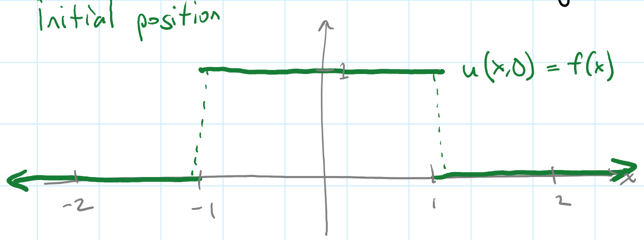
variable of integration

D'Alembert's Solution

(f) Initial conditions: $u(x,0) = f(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

$$\frac{\partial u}{\partial x}(x,0) = g(x) = 0 \qquad \text{(no initial velocity)}$$

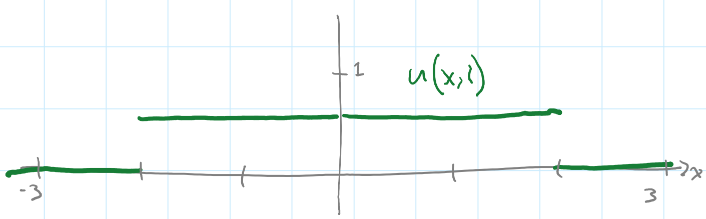
initial position



Since $g(x) = 0$, we have

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

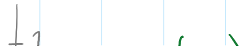
Let $c = 1$.

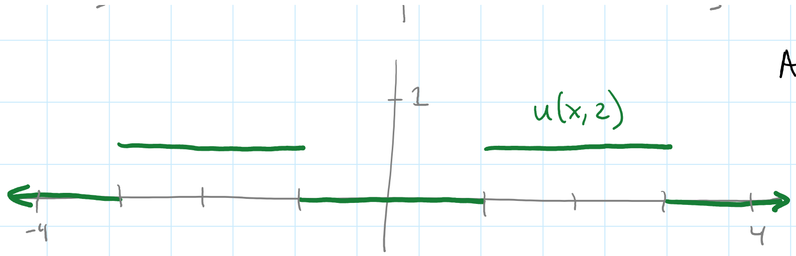


At time $t = 1$:

$$u(x,1) = \frac{1}{2}\left[\underbrace{f(x+1)}_{\text{shift left by 1}} + \underbrace{f(x-1)}_{\text{shift right by 1}}\right]$$

At time 2:





At time 2:

$$u(x,2) = \frac{1}{2}\left[f(x+2) + f(x-2)\right]$$

Mathematica animations: http://mlwright.org/teachin/math330f19/other/traveling_waves.nb

EXAMPLE: $f(x) = \cos(x)$

$g(x) = 0$

Now: $u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$

$$u(x,t) = \frac{1}{2}\left[\cos(x+ct) + \cos(x-ct)\right]$$

Two cosine waves, traveling in opposite directions, with speed c.

EXAMPLE: $f(x) = 0$

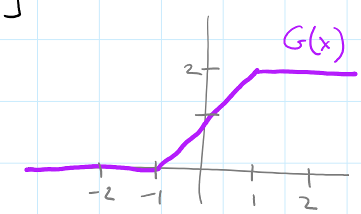
$$g(x) = \begin{cases} 1, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

String initially at rest, tap it with a hammer of width 2, centered at 0

Solution: $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau$

Let $c=1$. $u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} g(\tau)\,d\tau = \frac{1}{2}\left[G(x+t) - G(x-t)\right]$

where $G(x) = \begin{cases} 0 & \text{if } x < -1 \\ x+1 & \text{if } -1 \le x < 1 \\ 2 & \text{if } 1 \le x \end{cases}$



D'ALEMBERT'S SOLUTION WITH BOUNDARIES

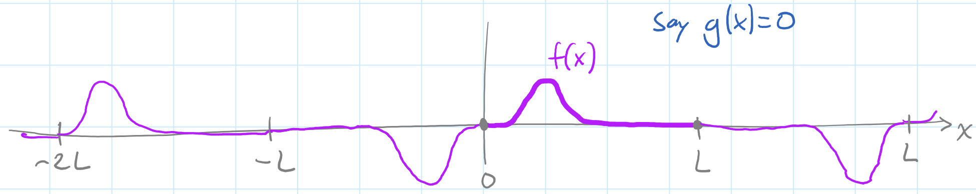
Suppose we have fixed (Dirichlet) boundary conditions: $u(0,t) = u(L,t) = 0$

Solution must be a Fourier sine series in x, so it's odd and $2L$-periodic.

Extend the initial position $f(x)$ and velocity $g(x)$ to be odd and $2L$-periodic:

Say $g(x)=0$

Say $g(x)=0$



Then D'Alembert's solution for the infinite string remains odd and 2L-periodic, and satisfies the boundary conditions $u(0,t)=u(L,t)=0$.

Note: Solution $u(x,t)$ is defined for all x, but only $0 \le x \le L$ is physically relevant.

But effects of initial conditions in the unphysical part will be felt in the physical part as traveling waves.

Animation by Peter Olver: https://www.youtube.com/watch?v=luRxgm_sk64

LINEAR PDEs

		category	Models
Heat Eq:	$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0$	"parabolic"	diffusion
Laplace Eq:	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	"elliptic"	equilibrium phenomena
Wave Eq:	$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$	"hyperbolic"	vibrations

Problem #3(f) on handout -- f is the “box” function, g is zero

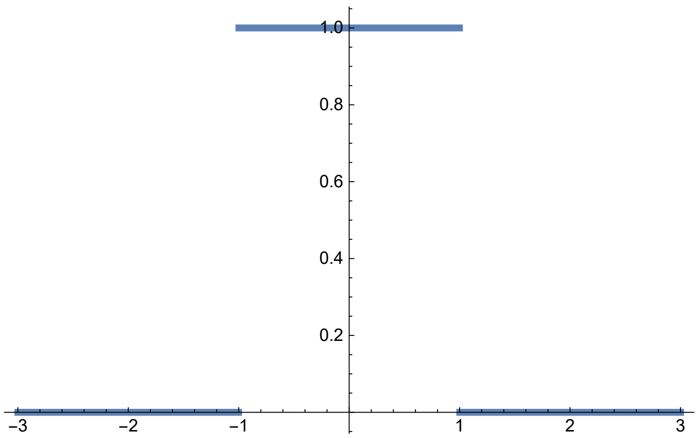
Initial position:

In[ ]:=
```
f1[x_] = Piecewise[{{1, -1 < x < 1}}]
```

Out[ ]=
$$\begin{cases} 1 & -1 < x < 1 \\ 0 & \text{True} \end{cases}$$

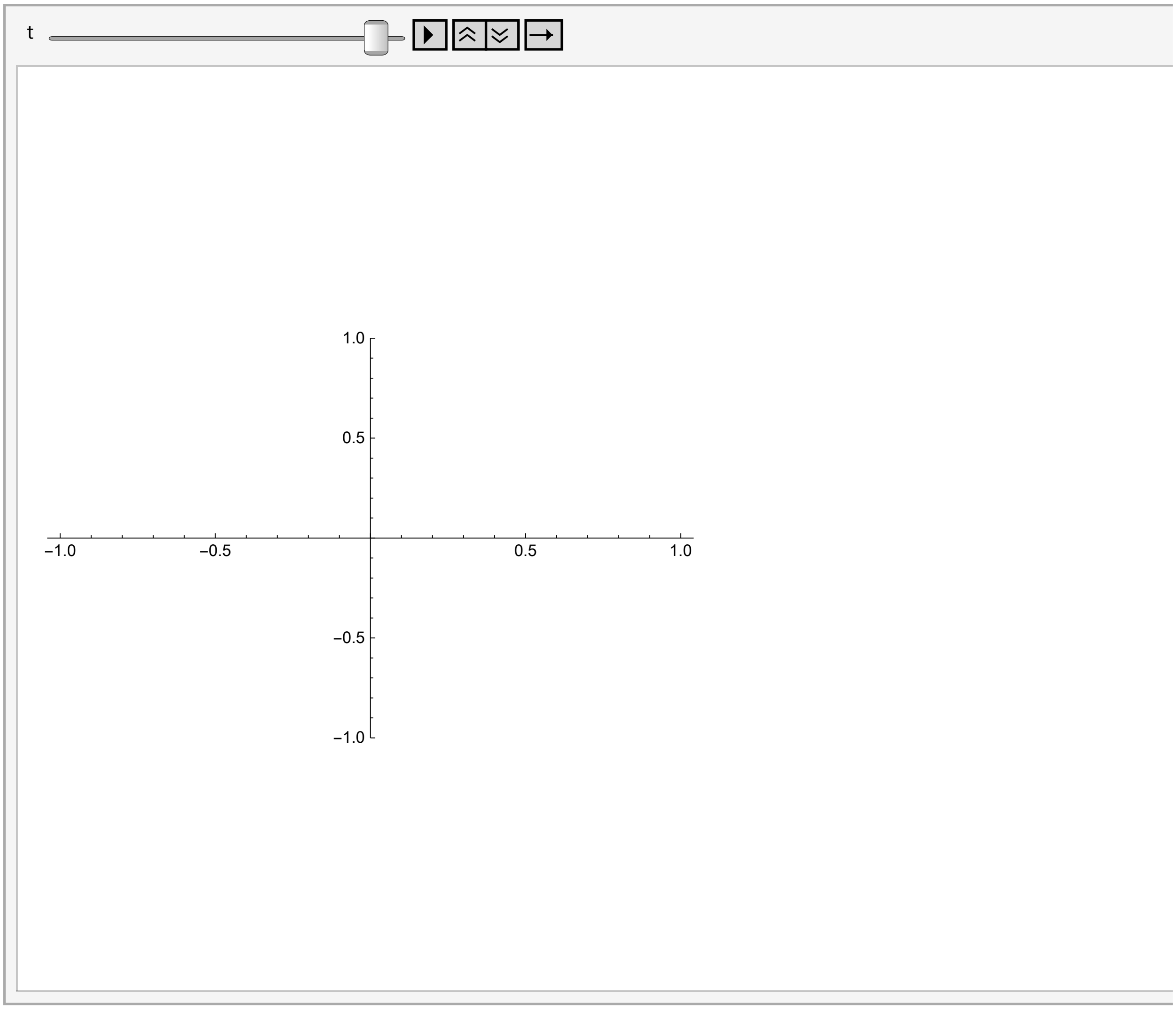
In[ ]:=
```
Plot[f1[x], {x, -3, 3}, PlotStyle → Thickness[0.01]]
```

Out[ ]=



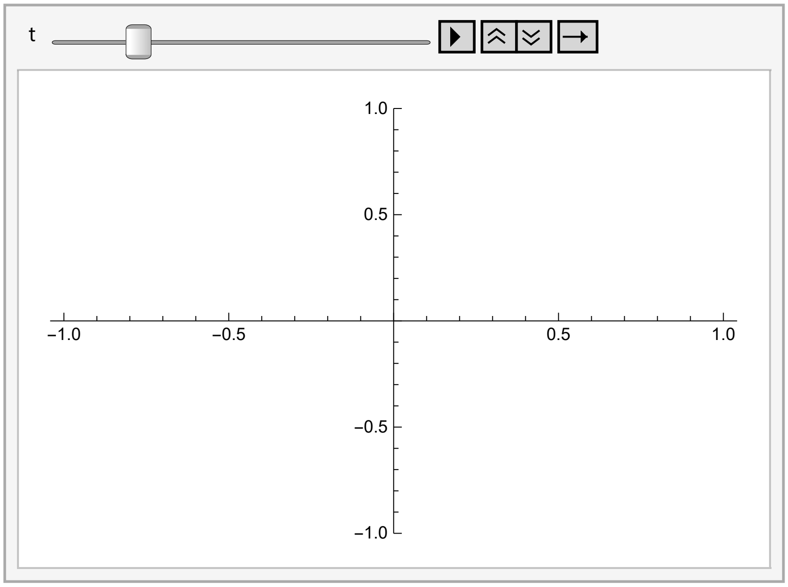
In[ ]:=
```
Animate[Plot[1/2 (f1[x - t] + f1[x + t]), {x, -6, 6}, PlotRange → {-1, 1},
  PlotStyle → Thickness[0.01]], {t, 0, 8}, AnimationRunning → False]
```

Out[ ]=



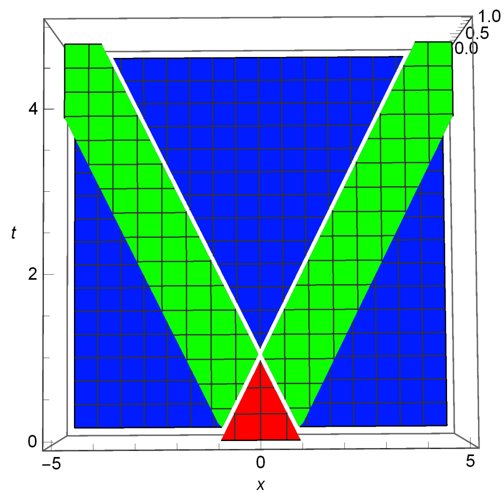
In[◦]:=
```
Animate[Plot[{1/2 (f1[x - t] + f1[x + t]), 1/2 f1[x - t], 1/2 f1[x + t]}, {x, -6, 6},
  PlotRange → {-1, 1}, PlotStyle → {Thickness[0.01], Thickness[0.005], Thickness[0.005]}],
 {t, 0, 8}, AnimationRunning → False]
```

Out[◦]=



In[◦]:=
```
Plot3D[1/2 (f1[x - t] + f1[x + t]), {x, -5, 5}, {t, 0, 5},
 ColorFunction → Function[{x, y, z}, Hue[.65 (1 - z)]], AxesLabel → Automatic]
```

Out[◦]=



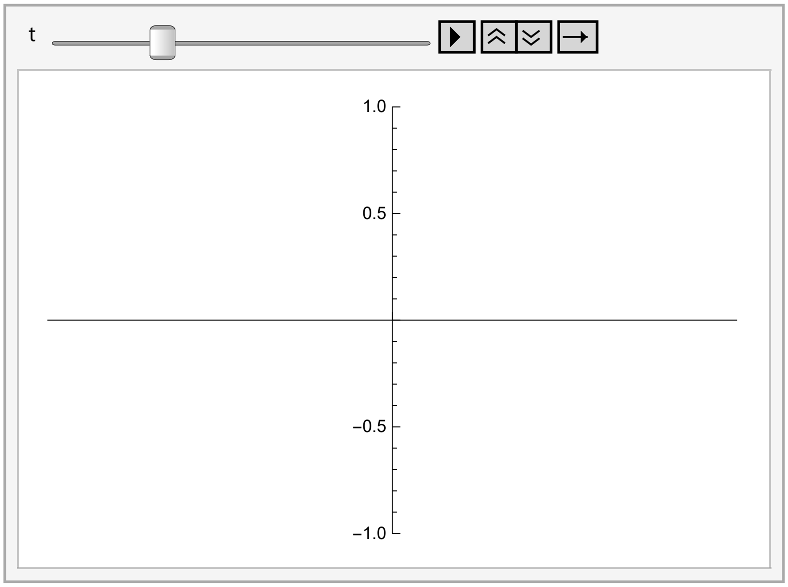
f is cosine, g is zero

In[◦]:=
```
f2[x_] = Cos[x]
```

Out[◦]=
```
Cos[x]
```

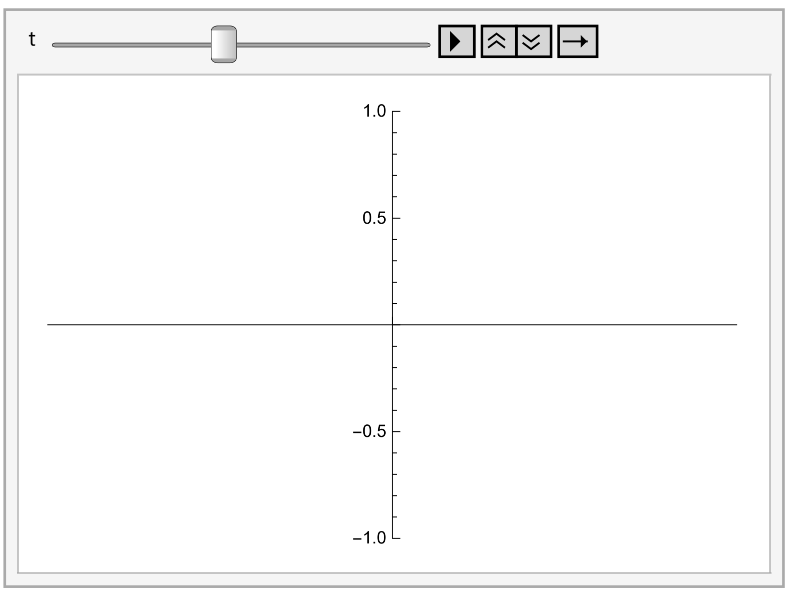
In[ ]:=
```
Animate[Plot[1/2 (f2[x - t] + f2[x + t]),
  {x, -4 Pi, 4 Pi}, PlotRange → {-1, 1}, PlotStyle → Thickness[0.01],
  Ticks → {Range[-4 Pi, 4 Pi, Pi], Automatic}], {t, 0, 8}, AnimationRunning → False]
```

Out[ ]=



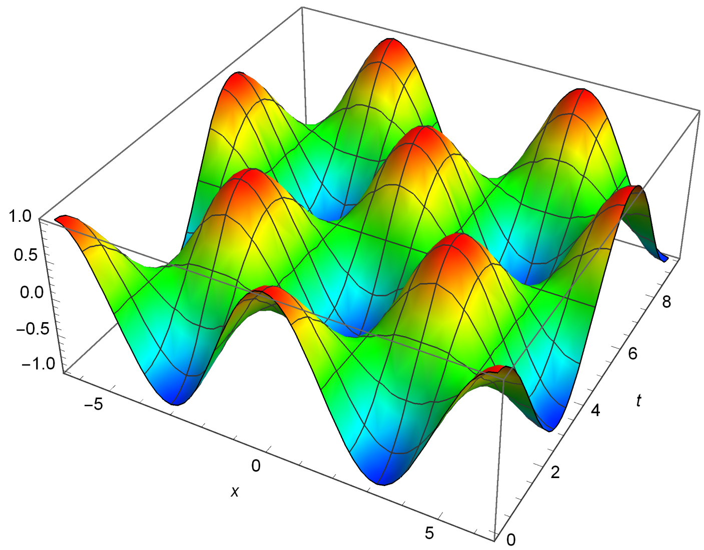
In[ ]:=
```
Animate[Plot[{1/2 (f2[x - t] + f2[x + t]), 1/2 f2[x - t], 1/2 f2[x + t]}, {x, -4 Pi, 4 Pi},
  PlotRange → {-1, 1}, PlotStyle → {Thickness[0.01], Thickness[0.005], Thickness[0.005]},
  Ticks → {Range[-4 Pi, 4 Pi, Pi], Automatic}], {t, 0, 6 Pi}, AnimationRunning → False]
```

Out[ ]=



In[◦]:=
```
Plot3D[1/2 (f2[x - t] + f2[x + t]), {x, -2 Pi, 2 Pi}, {t, 0, 3 Pi},
  ColorFunction → Function[{x, y, z}, Hue[.65 (1 - z)]], AxesLabel → Automatic]
```

Out[◦]=



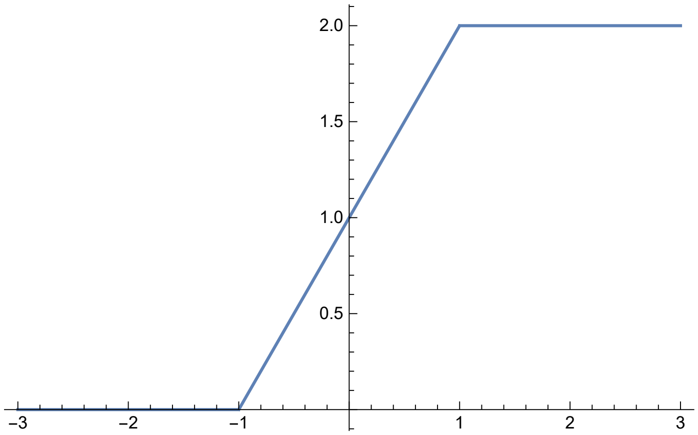
f is zero, g is the “ramp” function

In[◦]:=
```
G[x_] = Piecewise[{{x + 1, -1 < x < 1}, {2, x ≥ 1}}]
```

Out[◦]=
$$G(x)=\begin{cases} 1 + x & -1 < x < 1 \\ 2 & x \geq 1 \\ 0 & \text{True} \end{cases}$$

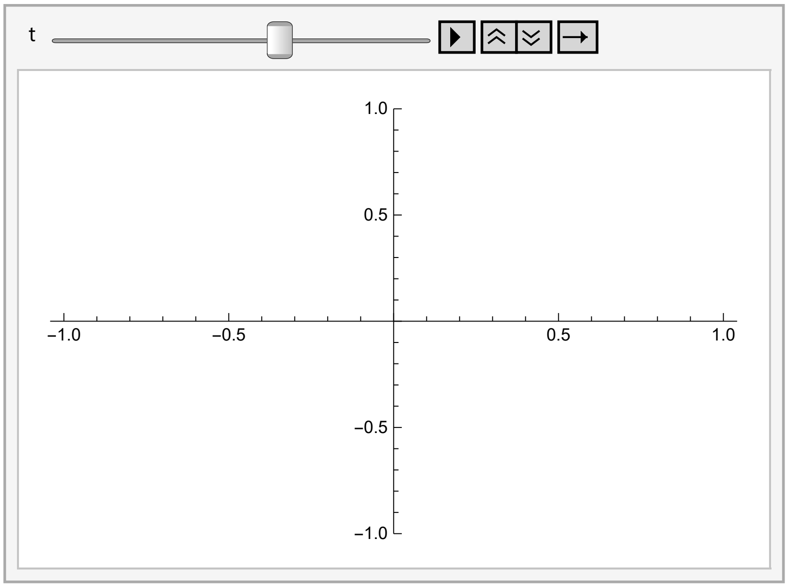
In[◦]:=
```
Plot[G[x], {x, -3, 3}]
```

Out[◦]=



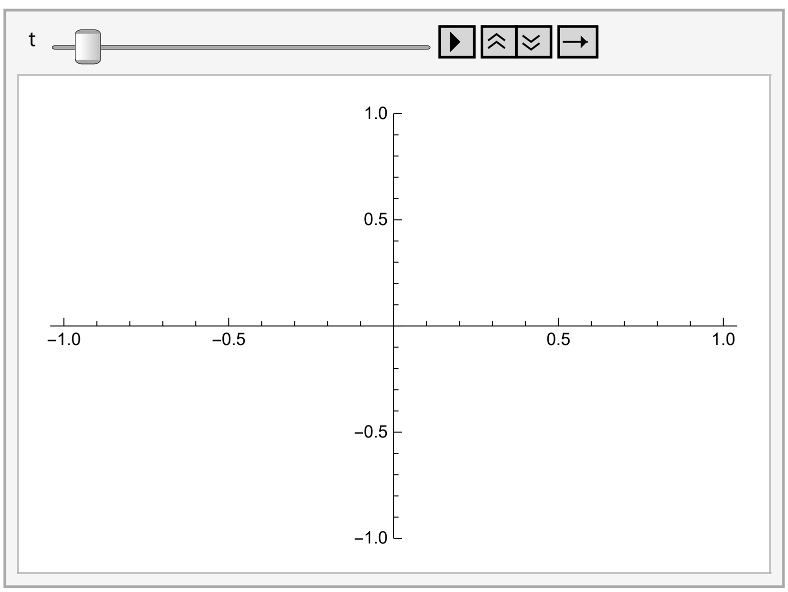
In[ ]:=
```
Animate[Plot[1/2 (G[x + t] - G[x - t]), {x, -6, 6}, PlotRange → {-1, 1},
  PlotStyle → Thickness[0.01]], {t, 0, 8}, AnimationRunning → False]
```

Out[ ]=



In[ ]:=
```
Animate[Plot[{1/2 (G[x + t] - G[x - t]), 1/2 G[x + t], -1/2 G[x - t]}, {x, -6, 6},
  PlotRange → {-1, 1}, PlotStyle → {Thickness[0.01], Thickness[0.005], Thickness[0.005]}],
 {t, 0, 8}, AnimationRunning → False]
```

Out[ ]=



In[ ]:=
```
Plot3D[1/2 (G[x + t] - G[x - t]), {x, -6, 6}, {t, 0, 8},
 ColorFunction → Function[{x, y, z}, Hue[.65 (1 - z)]], AxesLabel → Automatic]
```

Out[ ]=