RECALL: **S-L EQ:** $\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0, \quad a<x<b$

p, q, σ real-valued functions, $p(x)>0$, $\underbrace{\sigma(x)>0}_{\text{"weight function"}}$

REGULAR: $\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) = 0, \quad \beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) = 0$

LAST TIME:

2. $\phi'' + 4\phi' + 8\phi + \lambda\phi = 0, \quad \phi(0) = \phi(L) = 0$

We multiplied by e^{4x} to put this eq. in S-L form.

Solve: char. poly. $r^2 + 4r + (8+\lambda) = 0$ so $r = -2 \pm \sqrt{-\lambda-4}$

If $-\lambda-4<0$, then $\sqrt{-\lambda-4} = i\sqrt{\lambda+4}$

No nontrivial solutions for $-\lambda-4 \geq 0$

If $\underbrace{-\lambda-4<0}_{\lambda+4>0}$: $\phi(x) = Ae^{-2x}\cos\left(x\sqrt{\lambda+4}\right) + Be^{-2x}\sin\left(x\sqrt{\lambda+4}\right)$

Boundary: $\phi(0)=0 \Rightarrow 0 = Ae^0\cos(0) + 0$

$0 = A$

$\phi(L) = 0 \Rightarrow 0 = Be^{-2L}\sin\left(L\sqrt{\lambda+4}\right)$

We need $L\sqrt{\lambda+4} = n\pi$, or $\sqrt{\lambda+4} = \frac{n\pi}{L}$, or $\lambda = \left(\frac{n\pi}{L}\right)^2 - 4$

for $n = 1, 2, 3, \dots$

Verify S-L Theorems 1-5:

- Eigenvalues are real, and they form an infinite sequence with a smallest, but no largest, eigenvalue
- Eigenfunctions: $\phi_n(x) = e^{-2x}\sin\left(\frac{n\pi}{L}x\right)$, one eigenfunction per eigenvalue
- Completeness: $f \sim \sum_{n=1}^{\infty} a_n\phi_n(x) = \sum_{n=1}^{\infty} a_n e^{-2x}\sin\left(\frac{n\pi}{L}x\right)$

If $f(x) = \sum_{n=1}^{\infty} a_n e^{-2x}\sin\left(\frac{n\pi}{L}x\right)$, then $f(x)e^{2x} = \sum_{n=1}^{\infty} a_n\sin\left(\frac{n\pi}{L}x\right)$

and we can find a_n: $a_n = \int_0^L f(x)e^{2x}\sin\left(\frac{n\pi}{L}x\right)dx$

- Orthogonality: $\int_0^L \phi_n\phi_m\sigma\, dx = 0$ if $m \neq n$

Here, $\sigma(x) = e^{4x}$: $\int_0^L \left(\cancel{e^{-2x}}\sin\left(\frac{n\pi}{L}x\right)\right)\left(\cancel{e^{-2x}}\sin\left(\frac{m\pi}{L}x\right)\right)\cancel{e^{4x}}\,dx = 0$ if $m \neq n$

3. Heat equation with non-constant thermal properties.

Suppose $u(x,t) = \phi(x)\, h(t)$ and separation of variables produces

$$\frac{dh}{dt} = -\lambda h \qquad \text{and} \qquad \frac{d}{dx}\left[K_0(x)\frac{d\phi}{dx}\right] = -\lambda\, c(x)\rho(x)\,\phi$$

(a) $\frac{d}{dx}\left[K_0 \frac{d\phi}{dx}\right] + \lambda \underbrace{c\rho}\, \phi = 0$ and $\phi(0) = \phi(L) = 0$ ← Regular S-L equation

$p(x) = K_0(x),\quad q(x) = 0,\quad \sigma(x) = c(x)\rho(x)$

(b) Assume $\phi_n(x)$ are known for $n = 1, 2, 3, \ldots$

zero, since $\phi(0) = \phi(L) = 0$

Then, the Rayleigh Quotient gives: $\lambda_n = \dfrac{-K_0\,\phi\,\frac{d\phi}{dx}\Big|_0^L + \int_0^L \left[K_0\left(\frac{d\phi}{dx}\right)^2 - 0\right]dx}{\int_0^L \phi^2 c\rho\, dx}$

Assume: $K_0(x) > 0$ for all x

$c(x) > 0$ " "

$\rho(x) > 0$ " "

$$\lambda_n = \frac{\int_0^L K_0\left(\frac{d\phi}{dx}\right)^2 dx}{\int_0^L \phi^2 c\rho\, dx} > 0$$

eigenvalues are positive!

(c) Since $h(t) = c e^{-\lambda t}$, series solution is $\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n\, \phi_n(x)\, e^{-\lambda_n t}}$

exists by S-L theorems

(d) If $u(x,0) = f(x) = \underbrace{\sum_{n=1}^{\infty} a_n \phi_n(x)}$

Since eigenfunctions are orthogonal with respect to weight function $c\rho$:

$$\int_0^L f(x)\, \phi_m(x)\, c\rho\, dx = \sum_{n=1}^{\infty} \int_0^L a_n\, \phi_n(x)\, \phi_m(x)\, c\rho\, dx = 0 \quad \text{unless } m = n$$

$$\int_0^L f(x)\, \phi_m(x)\, c\rho\, dx = \int_0^L a_n \phi_m^2\, c\rho\, dx$$

Solve for a_n:

$$a_n = \frac{\int_0^L f\, \phi_m\, c\rho\, dx}{\int_0^L \phi_m^2\, c\rho\, dx}$$

§5.5

OPERATOR: a function on functions

example: $\boxed{L(y) = \frac{d}{dx}\left[p\frac{dy}{dx}\right] + qy}$

input to L is a function $y(x)$

example: If $y = x^2$:

$$L(x^2) = \frac{d}{dx}\left[p\frac{d}{dx}(x^2)\right] + qx^2$$
$$= \frac{d}{dx}\left[p\cdot 2x\right] + qx^2$$
$$= \frac{dp}{dx}\cdot 2x + 2p + qx^2$$

For this operator L, the S-L equation can be written

$$L(\phi) + \lambda\sigma\phi = 0.$$

LAGRANGE'S IDENTITY: Let u and v be any functions of x.

Consider: $L(u) = \frac{d}{dx}\left[p\frac{du}{dx}\right] + qu$ and $L(v) = \frac{d}{dx}\left[p\frac{dv}{dx}\right] + qv$

Then: $$uL(v) - vL(u) = u\cdot\frac{d}{dx}\left[p\frac{dv}{dx}\right] + \cancel{quv} - v\frac{d}{dx}\left[p\frac{du}{dx}\right] - \cancel{quv}$$

SAME!

Product Rule: $$\frac{d}{dx}\left[u\cdot p\frac{dv}{dx}\right] = \frac{du}{dx}\cdot p\frac{dv}{dx} + u\cdot\frac{d}{dx}\left(p\frac{dv}{dx}\right)$$

Subtract $$-\left(\frac{d}{dx}\left[v\cdot p\frac{du}{dx}\right] = \frac{dv}{dx}\cdot p\frac{du}{dx} + v\frac{d}{dx}\left(p\frac{du}{dx}\right)\right)$$

$$\frac{d}{dx}\left[u\cdot p\frac{dv}{dx}\right] - \frac{d}{dx}\left[v\cdot p\frac{du}{dx}\right] = 0 + u\cdot\frac{d}{dx}\left(p\frac{dv}{dx}\right) - v\frac{d}{dx}\left(p\frac{du}{dx}\right)$$

So: $$\boxed{uL(v) - vL(u) = \frac{d}{dx}\left[u\cdot p\frac{dv}{dx}\right] - \frac{d}{dx}\left[v\cdot p\frac{du}{dx}\right]}$$

LAGRANGE'S IDENTITY

Integrate to obtain:

$$\int_a^b \left(uL(v) - vL(u)\right)dx = \int_a^b \left(\frac{d}{dx}\left[u\cdot p\frac{dv}{dx}\right] - \frac{d}{dx}\left[v\cdot p\frac{du}{dx}\right]\right)dx$$

$$\int_a^b \left(uL(v) - vL(u)\right)dx = \left[u\cdot p\frac{dv}{dx} - v\cdot p\frac{du}{dx}\right]_a^b$$

$$\boxed{\int_a^b \left(uL(v) - vL(u)\right)dx = \left[p\left(u\cdot\frac{dv}{dx} - v\frac{du}{dx}\right)\right]_a^b}$$

GREEN'S IDENTITY

$$\int_a (uL(v) - vL(u))\,dx = \left[p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]_a$$ **IDENTITY**

For most boundary conditions that we encounter (e.g. Regular S-L boundaries), this is zero.

EXAMPLE: Let ϕ_m and ϕ_n be eigenfunctions of a regular S-L problem, corresponding to different eigenvalues. $(m \neq n)$

So:
$$\phi_n L(\phi_m) + \lambda_m \sigma \phi_m \phi_n = 0$$
$$-\left(\phi_m L(\phi_n) + \lambda_n \sigma \phi_n \phi_m = 0\right) \quad \text{subtract}$$

$$\phi_n L(\phi_m) - \phi_m L(\phi_n) + (\lambda_m - \lambda_n)\sigma \phi_n \phi_m = 0$$

Integrate:
$$\underbrace{\int_a^b \left(\phi_n L(\phi_m) - \phi_m L(\phi_n)\right) dx}_{\text{Regular S-L boundary conditions imply this is zero.}} = (\lambda_n - \lambda_m)\int_a^b \sigma \phi_n \phi_m \, dx$$

$$0 = (\lambda_n - \lambda_m)\int_0^b \sigma \phi_n \phi_m \, dx$$

Thus, either $\lambda_n = \lambda_m$, or ϕ_n and ϕ_m are orthogonal!