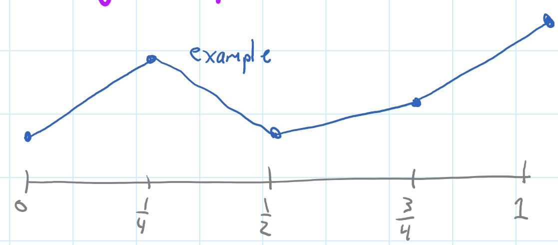
FINITE ELEMENT METHOD: 1-D EXAMPLE

$$-\frac{d}{dx}\left(K(x)\,\frac{du}{dx}\right) = f(x), \qquad 0 \le x \le 1, \qquad u(0)=u(1)=0$$

Models an elastic bar of non-uniform stiffness $K(x)$, with ends fixed, subject to an external force $f(x)$

IDEA: convert the ODE to a linear algebra problem!

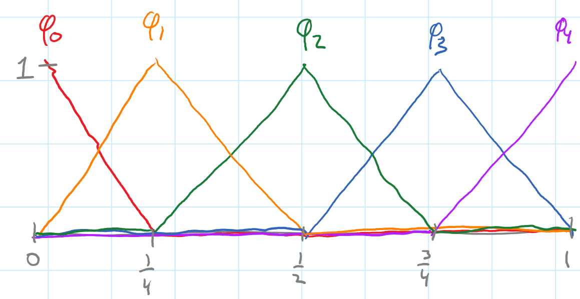
Partition the interval $[0,1]$ into n subintervals (here, $n=4$).



Look for an approximate solution that is piecewise-affine on the subintervals

"piecewise-linear"

Basis for the solution space consists of "hat" functions $\varphi_0, \varphi_1, \dots, \varphi_n$



General solution:

$$w(x) = c_0\varphi_0 + c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 \quad \leftarrow \text{"weak" solution}$$

Boundary conditions imply $c_0 = 0$, $c_4 = 0$. So $w(x) = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3$

We want $w(x)$ to satisfy:

$$\boxed{\int_0^1 -\frac{d}{dx}\left(K(x)\,\frac{dw}{dx}\right)v(x)\,dx = \int_0^1 f(x)\,v(x)\,dx \quad \text{for all } v(x) = d_1\varphi_1 + d_2\varphi_2 + d_3\varphi_3}$$

$\int u\,dv = uv - \int v\,du$

Integrate by parts:

$$\int_0^1 K(x)\,\frac{dw}{dx}\cdot\frac{dv}{dx}\,dx = \int_0^1 f(x)\,v(x)\,dx$$

$$\int_0^1 K(x)\left(c_1\varphi_1' + c_2\varphi_2' + c_3\varphi_3'\right)\left(d_1\varphi_1' + d_2\varphi_2' + d_3\varphi_3'\right)dx = \int_0^1 f(x)\left(d_1\varphi_1 + d_2\varphi_2 + d_3\varphi_3\right)dx$$

$$\sum_{i,j=1}^{3} c_i d_j \int_0^1 K(x)\,\varphi_i'\,\varphi_j'\,dx = \sum_{i=1}^{3} d_i \int_0^1 f(x)\,\varphi_i\,dx$$

Matrix equation: $\boxed{c^T K d = b^T d}$

where $c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$, $K = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix}$, $d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$, $b = \begin{pmatrix} \int_0^1 f\,\varphi_1\,dx \\ \int_0^1 f\,\varphi_2\,dx \\ \int_0^1 f\,\varphi_3\,dx \end{pmatrix}$

$$k_{ij} = \int_0^1 K(x)\,\varphi_i'\,\varphi_j'\,dx$$

Want $c^T K d = b^T d$ to be true for all d

$$(c^T K - b^T)\,d = 0$$

$(Kc - b)^T d = 0 \leftarrow$ this holds for all d iff $Kc - b = 0$

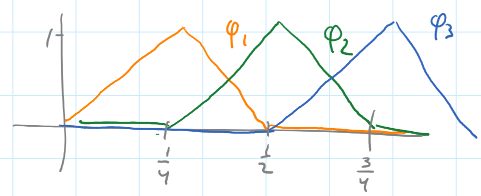
or $Kc = b$

Thus, we want to find $c = K^{-1} b$.

EXAMPLE: $K(x) = 1$, $f(x) = 1$ ODE: $-\dfrac{d^2u}{dx^2} = 1.$

$$k_{11} = \int_0^1 1\,(\varphi_1')^2\,dx = \int_0^{\frac{1}{2}} (4)^2\,dx = 8$$

similarly, $k_{22} = k_{33} = 8$



$$k_{12} = \int_0^1 1\,\varphi_1'\,\varphi_2'\,dx = \int_{\frac{1}{4}}^{\frac{1}{2}} (4)(-4)\,dx = -4$$

also, $k_{21} = k_{32} = k_{23} = -4$

$$k_{13} = \int_0^1 1\,\varphi_1'\,\varphi_3'\,dx = 0$$

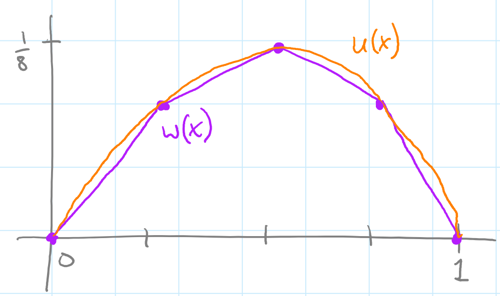
also, $k_{31} = 0$

so $K = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}$

Also: $\int_0^1 1\,\varphi_1(x)\,dx = \frac{1}{4}$ so $b = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$

Solution:

$$c = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}^{-1} \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 3/32 \\ 1/8 \\ 3/32 \end{bmatrix}$$



Exact solution: $u(x) = \frac{1}{2}(x - x^2)$

WORKSHEET:

$$\int_0^{\frac14} \_\_ \, dx + \int_{\frac14}^{\frac12} \_\_ \, dx = 16(s_0+s_1)$$

$$k_{11} = \int_0^1 (x+1)(\varphi_1')^2 \, dx = \int_0^{\frac12}(x+1)(4)^2 \, dx = 16\left(\frac{x^2}{2}+x\right)\Big|_0^{\frac12} = 16\left(\frac18+\frac12\right) = 16\left(\frac58\right) = 10$$

$$s_j = \int_{\frac14 j}^{\frac14(j+1)} (x+1)\, dx = \frac{x^2}{2}+x\,\Big|_{\frac14 j}^{\frac14(j+1)} = \frac{\left(\frac{(j+1)}{4}\right)^2}{2} + \frac{j+1}{4} - \left(\frac{j^2}{4^2\cdot 2}+\frac{j}{4}\right) = \frac{j^2+2j+1}{32} + \frac{j+1}{4} - \left(\frac{j^2}{32}+\frac{j}{4}\right)$$

$$= \frac{2j+1}{32}+\frac14 = \frac{2j+9}{32}$$

$$k_{11} = \int_0^{\frac14}(x+1)(4)^2\, dx = 16(s_0+s_1)$$

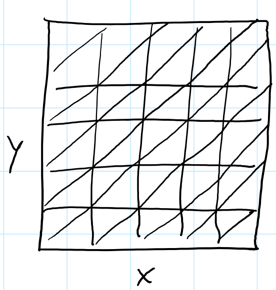
$$k_{22} = \int_{\frac14}^{\frac34}(x+1)\underset{\uparrow\ \varphi_2'}{(4^2)}\, dx = 16(s_1+s_2)$$

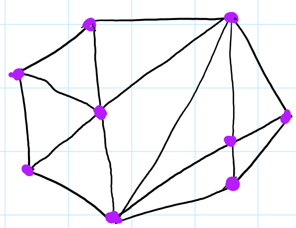
$$k_{12} = \int_{\frac14}^{\frac12}(x+1)\underset{\uparrow\ \varphi_1'}{(4)}\underset{\uparrow\ \varphi_2'}{(-4)}\, dx = -16\, s_1$$

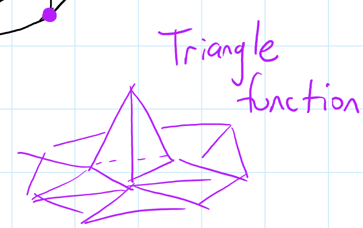
$$K = 16\begin{bmatrix} s_0+s_1 & -s_1 & 0 \\ -s_1 & s_1+s_2 & -s_2 \\ 0 & -s_2 & s_2+s_3 \end{bmatrix}$$

$$b = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

2 VARIABLES?







Computation for Worksheet Problem

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In[1]:= s[j_] := (2 j + 9) / 32
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In[2]:= K = {{s[0] + s[1], -s[1], 0}, {-s[1], s[1] + s[2], -s[2]}, {0, -s[2], s[2] + s[3]}} * 16
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Out[2]= $\left\{\left\{10, -\frac{11}{2}, 0\right\}, \left\{-\frac{11}{2}, 12, -\frac{13}{2}\right\}, \left\{0, -\frac{13}{2}, 14\right\}\right\}$

```
In[3]:= c = Inverse[K].{1/4, 1/4, 1/4}
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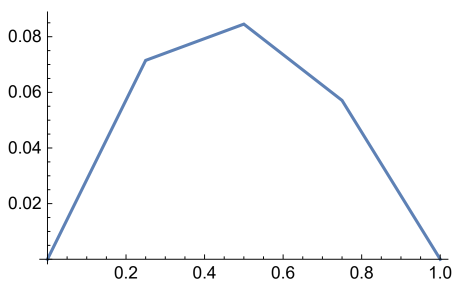
Out[3]= $\left\{\frac{159}{2224}, \frac{47}{556}, \frac{127}{2224}\right\}$

```
In[4]:= c = Append[Prepend[c, 0], 0]
```

Out[4]= $\left\{0, \frac{159}{2224}, \frac{47}{556}, \frac{127}{2224}, 0\right\}$

```
In[5]:= lp = ListPlot[c, Joined → True, DataRange → {0, 1}]
```

Out[5]=

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In[7]:= ep = Plot[-x + Log[1 + x] / Log[2], {x, 0, 1}, PlotStyle → Orange]
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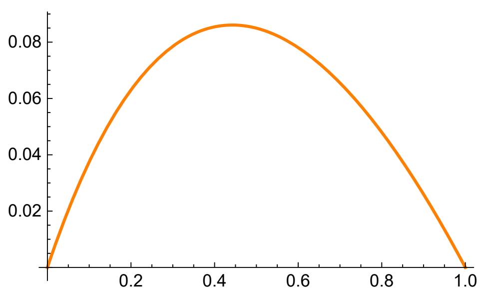
Out[7]=

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In[ ]:= Show[lp, ep]
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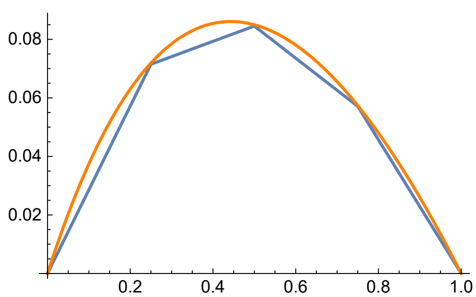
Out[ ]=



Finite Element Method

Math 330

Consider the boundary value problem

$$-\frac{d}{dx}\left((x+1)\frac{du}{dx}\right) = 1, \qquad 0 \le x \le 1, \qquad u(0) = u(1) = 0.$$

This equation models the deformation of a nonuniform rod with fixed ends and stiffness given by $\kappa(x) = x + 1$. We will find an approximate solution using the finite element method.

1. Partition the rod into $n = 4$ equal-length subintervals. Let $0 = x_0 < x_1 < x_2 < x_3 < x_4 = 1$ be the endpoints of the subintervals. Let $\phi_j(x)$ be the piecewise-linear function with $\phi_j(x_j) = 1$ and $\phi_j(x_k) = 0$ for $j \neq k$. Sketch the graphs of $\phi_0, \phi_1, \ldots, \phi_4$.

2. The functions $\phi_0, \ldots, \phi_4$ form a basis for our space of approximate solutions. That is, we are looking for an approximate solution of the form

$$w(x) = c_0\phi_0(x) + \cdots + c_4\phi_4(x),$$

for some coefficients $c_0, c_1, \ldots, c_4$. Explain why $c_0 = 0$ and $c_4 = 0$.

3. We find our approximate solution by solving the linear system $\mathbf{Kc} = \mathbf{b}$, where $\mathbf{c} = (c_1, c_2, c_3)^T$, $\mathbf{K}$ is a 3×3 matrix whose entry in row i column j is

$$k_{ij} = \int_0^1 (x+1)\phi_i'(x)\phi_j'(x)\ dx,$$

and $\mathbf{b} = (b_1, b_2, b_3)^T$ with $b_i = \int_0^1 \phi_i(x)\ dx$.

Evaluate the integrals to write $\mathbf{K}$ as a matrix of *numbers.* (No variables or integrals, please!)

Similarly, write $\mathbf{b}$ as a vector of *numbers.*

4. Solve for $\mathbf{c}$ by computing $\mathbf{c} = \mathbf{K}^{-1}\mathbf{b}$. (Use technology.)

5. Sketch your approximate solution. Does it look reasonable?

6. Show that the explicit solution is

$$u(x) = -x + \frac{\log(1+x)}{\log(2)}.$$

(You could find this solution by integrating twice, but for now simply verify that it satisfies the ODE and the boundary conditions.) Plot this solution, and compare it to the approximate solution you obtained by the finite element method.