Exam 2 Name: ______________________________

Math 330, Fall 2019

Due Tuesday, November 26 at **8:00am**

Instructions:

- Solve any 4 of the following 5 problems.
- You may use your textbook, your notes, the course web site, computational technology (e.g., *Mathematica*), homework assignments/solutions, and LaTeXhelp resources.
- *Do not consult other sources, web sites, or people other than the professor.*
- Type your solutions in LaTeX. If you use technology to compute something, indicate what you computed. Make sure to explain your solutions clearly, check your work, and proofread.
- *Make sure you attend to the pledge that at the end of this exam.*

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1. Suppose that $f(x)$ and df/dx are piecewise smooth. Prove that the Fourier series of $f(x)$ can be differentiated term by term if the Fourier series of $f(x)$ is continuous.

2. Consider the heat equation for a disk of radius a with constant thermal properties and a circularly symmetric temperature distribution (i.e., temperature $u(r,t)$ does not depend on the angle):

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \quad 0 < r < a, t > 0\\
&u(0,t) \text{ is bounded}\\
&u(a,t) = 0\\
&u(r,0) = f(r)
\end{aligned}$$

(a) Separate variables and show that the spatial equation is in Sturm-Liouville form. What are p, q, and σ?

(b) Prove that the eigenfunctions of this Sturm-Liouville problem are orthogonal.

(c) Solve for $u(r,t)$ assuming that the eigenfunctions $\phi_n(r)$ are known (and therefore the corresponding λ_n are known). Write down an expression for the coefficients.

(d) What is the dominant (i.e., largest) term in $u(r,t)$ for large t? What is $\lim_{t\to\infty} u(r,t)$?

3. Solve the following nonhomogeneous problem:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + e^{-2t}\sin(4\pi x)\\
&u(0,t) = 0\\
&u(1,t) = 0\\
&u(x,0) = x - x^2
\end{aligned}$$

You may assume $2 \neq k4^2\pi^2$. *Hint*: Use the method of eigenfunction expansion (Section 3.4).

4. Consider the PDE with boundary and initial conditions:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le 1, \quad t > 0 \\
&u(0,t) = u(1,t) = 0 \\
&u(x,0) = \begin{cases} \frac{x}{2}, & 0 \le x \le \frac{1}{2} \\ \frac{1}{2} - \frac{x}{2}, & \frac{1}{2} < x \le 1 \end{cases} \\
&\frac{\partial u}{\partial t}(x,0) = 0
\end{aligned}$$

(a) What physical situation is modeled by this PDE with the given boundary/initial conditions?

(b) Find the solution $u(x,t)$.

(c) Sketch a graph of your solution. Explain why your solution is reasonable for the physical situation you identified in part (a).

5. Consider the eigenvalue problem

$$\begin{aligned}
&\frac{d^2 y}{dx^2} + \lambda y = 0, \qquad 0 < x < 3 \\
&y(0) = 0 \\
&y(3) + y'(3) = 0
\end{aligned}$$

Based on what you've learned this semester, say as much as you can about the eigenvalues and eigenfunctions for this problem.

St. Olaf Honor Pledge: I pledge my honor that on this examination I have neither given nor received assistance not explicitly approved by the professor and that I have seen no dishonest work.

Signed: ______________________________

☐ I have intentionally not signed the pledge. (Check the box if appropriate.)