ODE Review

Math 330

1. (a) What is a **linear** ordinary differential equation?

A linear ODE has the form $a_n(t)y^{(n)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = F(t)$ for some coefficient functions $a_0(t), \ldots, a_n(t)$ and some function $F(t)$.

(b) Give examples of two linear ODEs of different **orders**.

Answers will vary. For example:

$$y'(t) = 2t + 1 \qquad \text{and} \qquad \frac{d^2y}{dt^2} + 2t\frac{dy}{dt} - y(t) = 0$$

(c) Do your examples have **constant coefficients** or **variable coefficients**?

If $a_0(t), \ldots, a_n(t)$ are constants, then the ODE has constant coefficients; otherwise it has variable coefficients. My first example in part (b) has constant coefficients, while my second example has variable coefficients.

(d) Are your examples **homogeneous** or **nonhomogeneous**?

If $F(t) = 0$, then the equation is homogenous; otherwise it is nonhomogeneous. My first example in part (b) is nonhomogeneous, while my second example is homogeneous.

2. Find two linearly independent solutions $y(t)$ to the differential equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = 0.$$

Describe the behavior of your solutions as $t \to \infty$ and $t \to -\infty$.

The characteristic equation is $r^2 + r - 6 = 0$, which has distinct real roots $r = -3$ and $r = 2$. This implies that there are linearly independent solutions $y_1(t) = e^{-3t}$ and $y_2(t) = e^{2t}$.

The solution $y_1(t) = e^{-3t}$ goes to ∞ as $t \to -\infty$ and to 0 as $t \to \infty$.

The solution $y_2(t) = e^{2t}$ goes to 0 as $t \to -\infty$ and to ∞ as $t \to \infty$.

3. Consider the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0.$$

(a) Find the general solution $y(t)$ to the differential equation.

The characteristic equation is $r^2 + 2r + 1 = 0$, which has a repeated root $r = -1$. This implies that there are linearly independent solutions $y_1(t) = e^{-t}$ and $y_2(t) = te^{-t}$. The general solution is

$$y(t) = c_1e^{-t} + c_2te^{-t}.$$

(b) Solve the initial value problem (IVP) given by the differential equation above and the initial conditions $y(0) = 1$ and $y'(0) = -1$.

The initial condition $y(0) = 1$ implies that $1 = c_1$, so $y(t) = e^{-t} + c_2te^{-t}$. Thus $y'(t) = -e^{-t} + c_2(1-t)e^{-t}$.

The initial condition $y'(0) = -1$ then implies that $-1 = -1 + c_2$, so $c_2 = 0$.

Therefore the particular solution to the IVP is

$$y(t) = e^{-t}.$$

4. Consider the differential equation

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 5y = 0.$$

(a) Find the general solution $y(t)$ to the differential equation.

The characteristic equation is $r^2 - 4r + 5 = 0$, which has complex roots $r = 2 \pm i$. This implies that there are linearly independent solutions $y_1(t) = e^{2t}\sin(t)$ and $y_2(t) = e^{2t}\cos(t)$. The general solution is

$$y(t) = c_1 e^{2t}\sin(t) + c_2 e^{2t}\cos(t).$$

(b) Solve the boundary value problem (BVP) given by the differential equation above and the boundary values $y(0) = 0$ and $y(2) = 1$.

The boundary condition $y(0) = 0$ implies $0 = c_1 \cdot 0 + c_2 \cdot 1$, so $c_2 = 0$.

The boundary condition $y(2) = 1$ then implies $1 = c_1 e^4 \sin(2)$, so $c_1 = \frac{1}{e^4\sin(2)} \approx 0.0201\ldots$.

Thus, the particular solution to the BVP is

$$y(t) = \frac{1}{e^4\sin(2)} e^{2t}\sin(t).$$

5. Find a solution $y(t)$ to the differential equation:

$$\frac{dy}{dt} + 2ty = t$$

One method: separation of variables

Move the $2ty$ to the right side of the differential equation and factor to obtain

$$\frac{dy}{dt} = t(1-2y).$$

Separate variables and integrate as

$$\int \frac{dy}{1-2y} = \int t\ dt.$$

Evaluating the integrals, we obtain

$$-\frac{1}{2}\ln|1-2y| = \frac{1}{2}t^2 + C.$$

Multiply both sides by -2, exponentiate to get rid of the log, and collect the constants:

$$1 - 2y = C' e^{-t^2}.$$

Solving for y, we find

$$y(t) = C'' e^{-t^2} + \frac{1}{2}.$$

Another method: integrating factor (For a review of the integrating factor method, see this video.)

The integrating factor is $\mu(t) = e^{\int 2t\ dt} = e^{t^2}$. Multiplying both sides of the ODE by this integrating factor, we obtain

$$\frac{dy}{dt}e^{t^2} + 2te^{t^2}y = te^{t^2}.$$

The left side of this equation is the derivative of a product; specifically

$$\frac{d}{dt}\left(ye^{t^2}\right) = te^{t^2}.$$

Integrating both sides, we obtain

$$ye^{t^2} = \int te^{t^2}\ dt = \frac{1}{2}e^{t^2} + C.$$

The solution is then

$$y(t) = \frac{1}{2} + Ce^{-t^2}.$$

6. Find a homogeneous linear ODE whose general solution is:

$$y(t) = c_1 e^{-t} + c_2 e^{2t} + c_3 t e^{2t}$$

We recognize that this solution arises from a homogeneous linear ODE whose characteristic equation has a root at $r = -1$ and a repeated root (multiplicity 2) at $r = 2$. That is, the characteristic equation must be

$$(r+1)(r-2)^2 = r^3 - 3r^2 + 4.$$

The ODE must then be

$$\frac{d^3y}{dy^3} - 3\frac{d^2y}{dt^2} + 4y = 0.$$