1. Consider the differential equation $\frac{\partial u}{\partial t} = 0$.

(a) If $u = u(t)$ is a function of t alone, then what are all solutions to the differential equation?

$\frac{du}{dt} = 0$ has solution $u(t) = c$ for any constant c.

INTEGRATING:

$$\int_0^t 0 \, dt = \int_0^t \frac{du}{dt} \, dt$$

$$c = u(t)$$

CONSIDER:

$$u(t) = \begin{cases} 1 & \text{if } t > 0 \\ -1 & \text{if } t < 0 \end{cases}$$

satisfies $\frac{du}{dt} = 0$ on the domain $(-\infty, 0) \cup (0, \infty)$

(b) If $u = u(t, x)$ then what are all solutions to the differential equation?

$u(t,x) = f(x)$ for some function $c(x)$

INTEGRATE WITH RESPECT TO t:

$$0 = \int_0^t \frac{\partial u(s,x)}{\partial t} \, ds = u(t,x) - u(0,x)$$

Thus $u(t,x) = u(0,x)$ — some function of x alone (say $f(x)$)

(c) Are all solutions to this differential equation constant in t? That is, if $u(t, x)$ is a solution, does it follow that $u(t_1, x) = u(t_2, x)$ if $t_1 \neq t_2$?

No — if the domain is disconnected, then $u(t,x)$ may take a different value on each connected component of the domain.

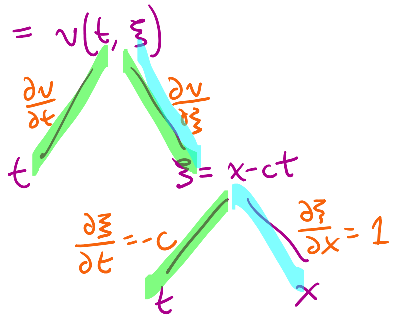
EXAMPLE:

$$u(t,x) = \begin{cases} x & \text{if } t > 0 \\ -x & \text{if } t < 0 \end{cases}$$

Satisfies $\frac{\partial u}{\partial t} = 0$ on the domain $\{(t,x) \in \mathbb{R}^2 \mid t \neq 0\}$

2. Consider the transport equation $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$, where c is some constant.

(a) Introduce the characteristic variable $\xi = x - ct$. That is, $u(t,x) = v(t, x - ct) = v(t,\xi)$. Use the multivariable chain rule to explain why $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} - c\frac{\partial v}{\partial \xi}$ and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi}$.



differentiate w.r.t. t:

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial \xi}\cdot\frac{\partial \xi}{\partial t} = \frac{\partial v}{\partial t} - c\frac{\partial \xi}{\partial t}$$

differentiate w.r.t. x:

$$\frac{\partial v}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} = \frac{\partial v}{\partial \xi}\cdot 1 = \frac{\partial v}{\partial \xi}$$

(b) How does the transport equation simplify when expressed in terms of the characteristic variable? What functions v satisfy this equation?

Transport equation: $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$

Substitute: $\left(\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x}\right) + c\left(\frac{\partial v}{\partial x}\right) = 0$

Simplify: $\frac{\partial v}{\partial t} = 0$ ← same as in problem #1

The solution may be any C^1 function $v(t,\xi) = f(\xi)$ of ξ alone.

(c) Transform your solution v back to the original variables t and x. Can you give a physical interpretation of this solution? (Hint: What are the roles of t, x, and c?)

We have: $u(t,x) = v(t, x-ct) = \underbrace{f(x-ct)}$

This is a traveling wave of unchanging shape moving with constant speed c.

3. Find the solution to the initial value problem $\frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} = 0$ with $u(0,x) = \frac{1}{1+x^2}$.

$c = 2$

Solution:

$$u(t,x) = \frac{1}{1+(x-2t)^2}$$

← This is a function of $x - 2t$ that satisfies $u(0,x) = \frac{1}{1+x^2}$.

4. Now consider the differential equation $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + au = 0$, where a is a positive constant and c is any constant.

(a) Introduce the change of variable $\xi = x - ct$ as before. How does this simplify the differential equation?

We will continue this next class.

(b) Multiply your equation by the integrating factor e^{at}. Show that $\frac{\partial}{\partial t}\left(e^{at}v\right) = 0$. What does this imply about $e^{at}v$?

(c) Let $f(\xi)$ be a C^1 function and suppose $e^{at}v = f(\xi)$. Solve for v and transform your solution back to the original variables t and x.

(d) What initial value problem have you now solved? Give a physical interpretation for your solution.

5. Find the solution to the initial value problem $\frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} + u = 0$ with $u(0, x) = \frac{1}{1+x^2}$.