Uniform transport: $\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$

speed of flow

characteristic lines have slope c in the tx-plane

$c = \frac{dx}{dt}$ if $x(t)$ is the position of a moving particle at time t

Transport with decay

4. Now consider the differential equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + au = 0$$

where a is a positive constant and c is any constant.

(a) Introduce the change of variable $\xi = x - ct$ as before. How does this simplify the differential equation?

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \frac{\partial v}{\partial t} \quad \text{as before.}$$

So the transport with decay equation becomes

$$\frac{\partial v}{\partial t} + av = 0 \qquad \text{note: } u(t,x) = v(t,\xi)$$

(b) Multiply your equation by the integrating factor e^{at}. Show that $\frac{\partial}{\partial t}\left(e^{at}v\right) = 0$. What does this imply about $e^{at}v$?

$$\frac{\partial v}{\partial t}e^{at} + ae^{at}v = 0$$

$$\frac{\partial}{\partial t}\left(e^{at}v\right) = 0$$

Thus, $e^{at}v(t,\xi)$ is const. w.r.t. t. That is $e^{at}v = f(\xi)$

(c) Let $f(\xi)$ be a C^1 function and suppose $e^{at}v = f(\xi)$. Solve for v and transform your solution back to the original variables t and x.

$$\Downarrow$$

$$v(t,\xi) = e^{-at}f(\xi)$$

thus: $u(t,x) = e^{-at}f(x-ct)$

(d) What initial value problem have you now solved? Give a physical interpretation for your solution.

wave with fixed velocity c and decay rate a

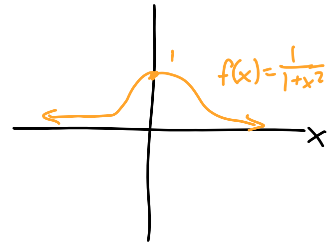
PDE: $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + au = 0$

IC: $u(0,x) = f(x)$

Gaussian: $f(x) = e^{-x^2}$

5. Find the solution to the initial value problem

$$\underbrace{\frac{\partial u}{\partial t} + \overset{c=2}{2}\frac{\partial u}{\partial x} + \overset{a=1}{u} = 0}_{\text{PDE}}, \qquad \underbrace{u(0,x) = \frac{1}{1+x^2}}_{\text{initial condition}}.$$



Solution:

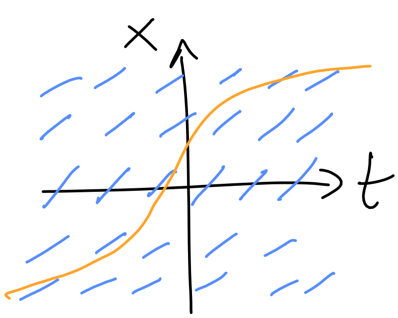
$$u(t,x) = \frac{e^{-t}}{1+(x-2t)^2}$$

nonuniform transport:

speed depends on position x

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = 0$$

now $\frac{\partial x}{\partial t} = c(x)$ is the speed at position x



example: $c(x) = \frac{1}{1+x^2}$

$$\frac{dx}{dt} = \frac{1}{1+x^2}$$

$\int (1+x^2)\,dx = \int dt$

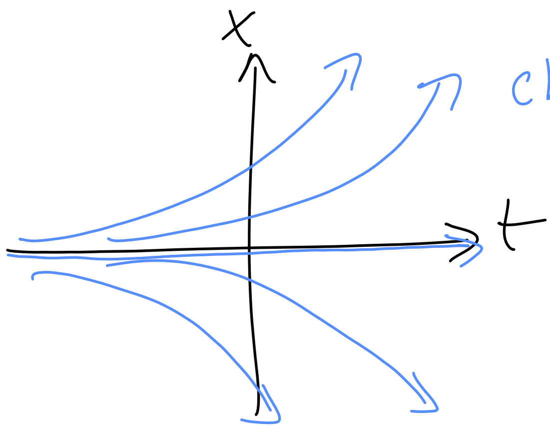
$x + \frac{1}{3}x^3 = t + K$

$\underbrace{\boxed{\frac{1}{3}x^3 + x - t}}_{\text{const}} = K$

6. Consider the nonuniform transport equation

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 0.$$

(a) Sketch some slope lines tangent to the characteristic curves for this equation. What is the shape of the characteristic curves?



characteristic curves

$\xi = \int \frac{dx}{c(x)} - t$

(b) The characteristic curves are given by what functions $x(t)$?

$\frac{dx}{dt} = x$

Characteristic variable: $\int \frac{dx}{c(x)} - t$

$\int \frac{dx}{\underbrace{x}_{c(x)}} = \int dt \implies e^{\ln|x|} = e^{t+K} \implies x(t) = \bar{K}e^{t}$

(c) Suppose $u(t,x)$ satisfies this differential equation. Describe in words how the graph of $u(t,x)$ changes as t increases. Optionally, you may assume an initial condition such as $u(0,x) = e^{-x^2}$.

All particles move away from the origin faster and faster over time

(d) Give an expression for the solution $u(t,x)$.