Separation of Variables

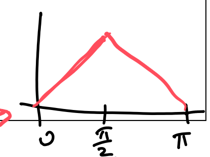
Math 330

Problem I: heat equation with homogeneous Dirichlet boundary conditions (i.e., zero temperature at endpoints)

PDE: $$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}$$

Boundary Conditions: $u(t,0) = 0$ and $u(t,\pi) = 0$

Initial Condition: $$u(0,x) = \begin{cases} x, & 0 < x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \pi \end{cases}$$



1. We will look for solutions of the form $u(t,x) = G(t)v(x)$, where $G = G(t)$ is a function of t only and $v = v(x)$ is a function of x only. *Plug this solution into the PDE and separate variables*: move everything that depends on t to the left side of the equation, and everything that depends on x to the right side. As good practice, keep the derivative expressions in the numerators, and group the γ constant with the G function.

$\frac{\partial u}{\partial t} = G'(t)v(x)$ and $\frac{\partial^2 u}{\partial x^2} = G(t)v''(x)$

$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}$ becomes $G'(t)v(x) = \gamma G(t)v''(x)$

Or: $$\frac{G'(t)}{\gamma G(t)} = \frac{v''(x)}{v(x)}$$

function of t | function of x alone

2. Each side of your new equation equals a constant, in fact the same constant. *Why is this?*

$$\frac{G'(t)}{\gamma G(t)} = \frac{v''(x)}{v(x)} = \text{some constant} = -\lambda$$

3. Denote the constant by $-\lambda$ (the negative is just for convenience later). Since each side of the equation equals $-\lambda$, this produces two ordinary differential equations, one in $G(t)$ and the other in $v(x)$. Write down these two ordinary differential equations. Simplify or rearrange them so that they look familiar enough to solve.

$G'(t) = -\lambda\gamma G(t)$ $\qquad$ $v''(x) = -\lambda v(x)$

exponential decr

4. Solve the time-dependent ODE for $G(t)$. Of the three options, $\boxed{\lambda > 0,}$ $\boxed{\lambda = 0,}$ or $\cancel{\lambda < 0}$, which ones seem the most physically relevant?

constant

$$G'(t) = -\lambda \gamma G(t)$$

has solution $\boxed{G(t) = c \cdot e^{-\lambda \gamma t}}$

Assume $c \neq 0$

5. The position-dependent ODE, along with the given boundary conditions, form a boundary value problem for $v(x)$. Find the general solution for each of the three cases, $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. Which cases yield a nontrivial solution?

$$v''(x) = -\lambda v(x)$$

boundary:

$u(t,0) = 0$

$G(t)v(0) = 0$

$v(0) = 0$

since $G \neq 0$

$u(t,\pi) = 0$

$v(\pi) = 0$

eigenvalues $\lambda_n = n^2$

eigenfunctions $v_n(x) = \sin(nx)$

for $n = 1, 2, 3, \ldots$

$u(t,x) = e^{-\gamma \lambda t} \sin(nx)$ for $n = 1, 2, 3, \ldots$

General solution to the PDE:

$$\boxed{u(t,x) = \sum_{n=1}^{\infty} b_n e^{-\gamma n^2 t} \sin(nx)}$$

6. You now have a general solution to Problem 1, using everything except the initial condition. Finally, use the initial condition to obtain a particular solution.

to be continued...

7. Use technology to plot your particular solution over time. (Set $\gamma = 1$.) Does it seem reasonable to you?

Problem II: heat equation with homogeneous Neumann boundary conditions (i.e., zero flux at endpoints)

PDE: $$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}$$

Boundary Conditions: $$\frac{\partial u}{\partial x}(t,0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(t,1) = 0$$

Initial Condition: $$u(0,x) = \begin{cases} x, & 0 < x \le \frac{1}{2} \\ 1 - x, & \frac{1}{2} < x < 1 \end{cases}$$

Work through the steps on the previous page to solve Problem II.