Topological Spaces

MATH 348

1. Find all possible topologies on $X = \{a, b\}$.

2. Find all topologies on $X = \{a, b, c\}$ that contain $\{a\}$ and $\{b\}$ as open sets.

3. Give an example of a topology on $\mathbb{R}$ with...

(a) ...exactly three open sets.

(b) ...exactly five open sets.

4. Is it possible for a topological space to have exactly one open set?

5. Let $X = \mathbb{R}$. Which of the following collections of sets form topologies on X?

(a) $\mathcal{T}_1 = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$

(b) $\mathcal{T}_2 = \{[a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$

(c) $\mathcal{T}_3 = \{S \subset \mathbb{R} \mid 0 \in S\} \cup \{\emptyset\}$

(d) $\mathcal{T}_4 = \{S \subset \mathbb{R} \mid S \text{ contains either 0 or 1}\} \cup \{\emptyset\}$

(e) $\mathcal{T}_5 = \{(a, b) \mid a, b \in \mathbb{R}\}$

6. Consider the space $X = \{a, b, c\}$ with topology $\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Which of the following functions are continuous in this topology?

(a) $f : X \to X$ defined by $f(a) = b$, $f(b) = c$, and $f(c) = a$

(b) $g : X \to X$ defined by $g(a) = b$, $g(b) = a$, and $g(c) = c$

7. State another continuous function $h : X \to X$ for the topological space in problem #6.

8. Consider the topology $\mathcal{T}_1$ in problem #5. Which of the following functions are continuous?

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$

(b) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = -x$

9. Give several examples of continuous functions on the topological space $\mathcal{T}_3$ in problem #5.

10. Let X be a topological space, and let $Y \subset X$ have the subspace topology.

(a) If A is open in Y, and Y is open in X, show that A is open in X.

(b) If A is closed in Y, and Y is closed in X, show that A is closed in X.

11. Consider the following subsets of $\mathbb{R}$ with the subspace topology.

(a) Let $K = \left\{\frac{1}{n} \in \mathbb{R} \mid n \in \mathbb{Z}_+\right\}$. Show that the subspace topology on K is the discrete topology.

(b) Let $K^* = K \cup \{0\}$. Show that the subspace topology on K is not the discrete topology.