Connectivity

MATH 348

1. Let $\mathcal{C} = \{(-n, n) \mid n \in \mathbb{Z}\}$.

(a) Does $\mathcal{C}$ cover $\mathbb{R}$?

(b) Does any finite subcollection of $\mathcal{C}$ cover $\mathbb{R}$?

(c) Does any finite subcollection of $\mathcal{C}$ cover $[0, 10]$?

2. Let $\mathcal{C} = \{(n, \infty) \mid n \in \mathbb{Z}\} \cup \{(-\infty, n) \mid n \in \mathbb{Z}\}$.

(a) Does $\mathcal{C}$ cover $\mathbb{R}$?

(b) Does any finite subcollection of $\mathcal{C}$ cover $\mathbb{R}$?

(c) Does any finite subcollection of $\mathcal{C}$ cover $[0, 10]$?

3. Let $\mathcal{C} = \{(x, x + 2^{-n}) \mid x \in \mathbb{R}, n \in \mathbb{Z}_+\}$.

(a) Does $\mathcal{C}$ cover $\mathbb{R}$?

(b) Does any finite subcollection of $\mathcal{C}$ cover $\mathbb{R}$?

(c) Does any finite subcollection of $\mathcal{C}$ cover $[0, 10]$?

4. Prove that any finite topological space is compact.

5. Let $A = \{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbb{Z}_+\right\}$. Is A compact in $\mathbb{R}$?

6. Is $(0, 1]$ compact as a subspace of $\mathbb{R}$?

7. Is $[0, 1]$ compact as a subspace of $\mathbb{R}$?

Important theorems:

- If T is a compact topological space and $f : T \to \mathbb{R}$ is a continuous function, then f is bounded.
- If $f : S \to T$ is a continuous map and S is compact, then the image of f is compact.
- A subspace T of $\mathbb{R}^n$ is compact if and only if T is closed (as a subset of $\mathbb{R}^n$) and bounded.

Challenge Problem: Let the set of rationals $\mathbb{Q}$ have the subspace topology from $\mathbb{R}$. Find a set $S \subset \mathbb{Q}$ that is closed and bounded but not compact.