29 October 2024

Definition: Two maps $f, g : S \to T$ are **homotopic** if there is a continuous function

$$F : S \times [0,1] \longrightarrow T$$

such that $F(s,0) = f(s)$ for all $s \in S$ and $F(s,1) = g(s)$ for all $s \in S$. In this case, F is a **homotopy** between f and g, and we write $f \simeq g$.

F interpolates continuously between f and g, as t goes from 0 to 1.

$F(s,t)$

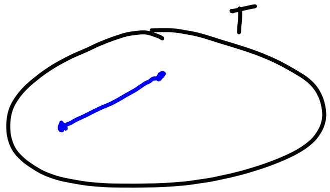
$s \in S$, $t \in [0,1]$ "time"

Proposition: Any two continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ are homotopic.

proof: use the straight-line homotopy

$$F(x,t) = (1-t)\, f(x) + t\, g(x)$$

for $x \in \mathbb{R}$ and $t \in [0,1]$



Homotopy

MATH 348

1. Suppose T is convex and S is any topological space. Why are every two maps $f, g : S \to T$ homotopic?

Since T is convex, we can se a straight-line ho

2. Give an example of spaces S and T with non-homotopic functions $f, g : S \to T$.

3. Justify each of the following to show that homotopy is an equivalence relation on functions between two given topological spaces:

(a) Homotopy is reflexive.

(b) Homotopy is symmetric.

(c) Homotopy is transitive.

4. Explain which of the following pairs of spaces are homotopy equivalent.

(a) The single point space $\{0\}$ and $\mathbb{R}$

(b) The closed interval $[0,1]$ and the single point space $\{0\}$

(c) The open interval $(0,1)$ and the single point space $\{0\}$

(d) The annulus $A = \{(x,y) \in \mathbb{R} \mid 1 \leq \sqrt{x^2+y^2} \leq 2\}$ and the circle S^1

(e) The punctured plane $\mathbb{R}^2 - \{(0,0)\}$ and the circle S^1

(f) S^1 and S^0