# Homework 6

MATH 348

due at 5pm on Tuesday, October 29, 2024

Solve the following problems and communicate your solutions clearly using complete sentences. Your proofs may rely on definitions and theorems stated in the text or given in class.

*Remember what the syllabus says about appropriate collaboration, and document what sources you use and what assistance you receive as you work on this homework.*

For this homework, you must type your solutions to all of the problems in LaTeX. You may include hand-drawn diagrams in your solutions. Make sure your solutions are easy to read, in order, and clearly labeled. Upload a single file containing your solutions to the Homework 6 assignment on Moodle.

Some of the problems will be graded in detail, and the rest will be graded for completion.

**1.** (10 points) In each of the following cases, describe the resulting quotient space. (A sketch may be helpful but is not a complete answer.)

(a) The disk with its boundary points identified with each other to form a single point.
(b) The interval $[0, 4]$, as a subspace of $\mathbb{R}$, with integer points identified with each other.
(c) The real line $\mathbb{R}$ with $[-1, 1]$ collapsed to a point.
(d) The plane $\mathbb{R}^2$ with the circle $S^1$ collapsed to a point.
(e) The sphere $S^2$ with the equator collapsed to a point.

**2.** (6 points) Let $X = \mathbb{R}$ and define an equivalence relation on $X$ by $x \sim y$ if $\lfloor x \rfloor = \lfloor y \rfloor$. Here, $\lfloor x \rfloor$ denotes the *floor* of $x$, the greatest integer less than or equal to $x$.

(a) Show that $\sim$ defines an equivalence relation on $\mathbb{R}$.
(b) What are the elements of the quotient space $X/\sim$?
(c) What are the open sets in the quotient space $X/\sim$?

**3.** (6 points) Let $X = \mathbb{R}^2$ and define an equivalence relation on $X$ by $(x, y) \sim (z, w)$ if

$$x^2 + y^2 = z^2 + w^2$$

(a) Show that $\sim$ defines an equivalence relation on $\mathbb{R}^2$.
(b) What are the elements of the quotient space $X/\sim$?
(c) What are the open sets in the quotient space $X/\sim$?

**4.** (4 points) Show that the topology defined on a quotient space (in Section 5.4 of our text) is really a topology. (That is, show that it satisfies the four axioms of a topology.)

**5.** (4 points) A surjective map $f : X \to Y$ is a **quotient map** if each $U \subset Y$ is open in $Y$ if and only if $f^{-1}(U)$ is open in $X$.

Is $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x$ a quotient map? Why or why not?

**6.** (4 points) A map $f : X \to Y$ is an **open map** if for each open set $U \subset X$, the set $f(U)$ is open in $Y$.

Show that a surjective, continuous map $f : X \to Y$ that is an open map is a quotient map.

**7.** (6 points) Exercise 3.40 on pages 110–111 in "Configuration Spaces and Phase Spaces" by Adams and Franzosa